Chapter 1 Section 2.

Note: This could also be rewritten in the style of 1.2 #1.

1. Air temperature usually decreases when the elevation increases. Sketch a possible graph of temperature as a function of distance driven for each of the following stories.

(a) I drove from the bottom to the top of a mountain.
(b) I drove from the top to the bottom of a mountain.
(c) I drove down a mountain, across a plateau, and up a higher mountain.
(d) I drove quickly down a mountain, across a plateau, turned around, and drove slowly back up the original mountain.

**ANSWER**

(a) As I drove up the mountain the temperature decreased with distance. A possible graph is

![Graph showing a linear decrease in temperature with distance](image1)

(b) As I drove down the mountain the temperature increased with distance. A possible graph is

![Graph showing a linear increase in temperature with distance](image2)
(c) As I drove down the mountain the temperature decreased with distance. As I drove across the plateau the temperature remained constant. As I drove up the mountain the temperature decreased with distance, ending lower than the temperature at the top of the first mountain. A possible graph is
(d) As I drove down the mountain the temperature decreased with distance. As I drove across the plateau the temperature remained constant. When I turned around and returned the temperature remained constant, but the distance continued to increase. As I drove up the mountain the temperature decreased with distance, ending at the same temperature as when I started. The speed I drove at does not affect the distance I drove, so the graph is symmetric about the turnaround point. A possible graph is

![Graph](image)

[ Straight line from (0, 1) to (4, 5), from (4, 5) to (6, 5), then from (6, 5) to (10, 1). No ticks. Label x-axis “Distance” and y-axis “Temperature” 0 ≤ x ≤ 10, 0 ≤ y ≤ 6.]

SHORT ANSWER
Not appropriate.
Chapter 1. Section with concavity (1.2?)

1. Two cylindrical barrels are identical except one has a bung hole through the flat side of the barrel while the other has it through the curved side. Each barrel is secured so the bung hole is uppermost, and they are then filled with liquid. Sketch a possible graph showing the amount of liquid in each barrel as a function of the height of the liquid. Explain how these graphs could be used to construct two dipsticks that determine the amount of liquid in each barrel.

ANSWER

For both barrels we note that when the height of the liquid is zero, the barrel is empty, so the graphs pass through the origin. We also note that the volume is always increasing, so the graph must be increasing.

The barrel with the bung on the flat side will be resting on the other flat side. Because the cross-section of the barrel is a circle and the same for all heights, the volume is a linear function of height. Thus, the graph will be a line through the origin.

The barrel with the bung on the curved side will be resting on the opposite curved side. The cross-section of the barrel at any height is a rectangle. However, these rectangles vary with height. They grow until the barrel is half full and then they shrink back the same way they grew. Thus, until the barrel is half full, when two equal volumes are added successively, the second additional height will be smaller than the first height. This means that the graph will be concave up until the barrel is half full. Then it will be concave down until it is completely full.
[To construct this graph, use the following data set, (it will run off the printed page, but is in the tex file),

<table>
<thead>
<tr>
<th>Height</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volume</td>
<td>0</td>
<td>6</td>
<td>16</td>
<td>30</td>
<td>45</td>
<td>61</td>
<td>79</td>
<td>98</td>
<td>117</td>
<td>137</td>
<td>157</td>
<td>177</td>
<td>197</td>
<td>216</td>
<td>235</td>
<td>253</td>
<td>269</td>
<td>288</td>
</tr>
</tbody>
</table>

and then join the points with a cubic spline smooth curve.]

Dipsticks can be constructed by reading off the heights for specific volumes and writing those volumes on a ruler at their corresponding heights.

SHORT ANSWER
Not appropriate.
Chapter 1. Section with concavity. (1.2?)

1. A barrel is filled with liquid and the graph shows the amount of liquid in the barrel (in gallons) as a function of the height of the liquid (in feet).

![Graph showing volume vs. height]

[Plot the function $125 - (5 - x)^3$]

(a) How tall is the barrel and how many gallons does it hold?

(b) Construct a dipstick for this barrel.

(c) When the barrel is full, is there more liquid in the top half or the bottom half of the barrel? Give a possible description of the barrel.

**ANSWER**

(a) Assuming that the graph stops when the barrel is full, it is 5 feet high, and holds about 125 gallons.

(b) From the graph we construct the following table

<table>
<thead>
<tr>
<th>Volume in gallons</th>
<th>0</th>
<th>20</th>
<th>40</th>
<th>60</th>
<th>80</th>
<th>100</th>
<th>120</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (approximate)</td>
<td>0.0</td>
<td>0.3</td>
<td>0.6</td>
<td>1.0</td>
<td>1.5</td>
<td>2.1</td>
<td>3.3</td>
</tr>
</tbody>
</table>

We would start with a stick longer than 5 feet, and then at 0.3 feet from the bottom mark “20 gallons”, at 0.6 feet from the bottom mark “40 gallons”, and so on.

(c) According to the graph, when the height is 2.5 feet there is about 110 gallons in the barrel, which only holds 125 gallons. There is much more in the bottom half than the top half.

There are 60 gallons in the barrel when the height is 1 foot, and about 100 when the height is two feet, that is, the extra foot gave us a net gain of $100 - 60 = 40$ gallons. This is generally true, so the more liquid we add the less
net gain, so the barrel must be getting narrower and narrower as we get higher and higher, rather like a cone.

SHORT ANSWER
(a) 5 feet, 125 gallons
(b) Not appropriate.
(c) Top. Cone-like.
Chapter 1 Section 3

1. Suppose you are given an instrument that looks like a thermometer. You are asked to find out what the instrument does and you place it in liquids at various known temperatures in degrees Fahrenheit and record the readings on the unknown instrument. The table shows the results of those experiments. What is the instrument?

\[\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{Temperature (°F)} & 30 & 50 & 70 & 90 & 110 & 130 \\
\text{Instrument Reading} & -1.1 & 10.0 & 21.1 & 32.2 & 43.3 & 54.4 \\
\hline
\end{array}\]

ANSWER

Since the instrument reading \(I\) increases by 11.1 units for every increase of 20°F in the temperature \(t\), the instrument reading could be a linear function with slope \(m = 0.555\). This would be consistent with the instrument measuring temperature, although not in Fahrenheit. If \(I\) is a linear function of \(t\), we have \(I = b + mt\) and \(m = 0.555\). To find \(b\), we substitute any point from the table, such as \(t = 50, I = 10\) into this equation: \(10 = b + (0.555)(50)\), so \(b = -17.75\). The equation of the line is \(I = -17.75 + 0.555t\). In Fahrenheit there are two natural temperatures to consider, 32 and 212, the freezing and boiling points of water. With \(t = 32\) we find \(I = 0.01\), and with \(t = 212\) we find \(I = 99.91\). To one decimal place, these are 0 and 100, the freezing and boiling points of water measured in degrees Celsius. Thus, the instrument is a thermometer calibrated in Celsius.

SHORT ANSWER

A thermometer calibrated in Celsius.

---

1 Adapted from J.G. Greeno, *Elementary Theoretical Psychology*, (Addison-Wesley, Massachusetts, 1968).
1. Suppose you are given an instrument that looks like it might measure weight. You are asked to find out what the instrument does. With nothing on the instrument you notice that it reads 46.0. You place various known weights in pounds on the instrument and record the readings on the unknown instrument. The table shows the results of those experiments.² What is the instrument?

**ANSWER**

Since the Instrument Reading \( I \) increases by 226.8 units for every increase of 0.5 pounds in the weight \( w \), the instrument reading could be a linear function with slope \( = 226.8/0.5 = 453.6 \). This would be consistent with the instrument measuring weight, although not in pounds. If \( I \) is a linear function of \( w \), we have \( I = b + mw \) and \( m = 453.6 \). To find \( b \), we substitute any point from the table, such as \( w = 0, I = 46 \) into this equation: \( 46 = b + (453.6)(0) \), so \( b = 46 \). The equation of the line is \( I = 46 + 453.6w \). However, when \( w = 0 \), the instrument reads 46, so this instrument has not been calibrated to weigh 0 pounds correctly. On this instrument, a 1 pound weight would register \( 46 + 453.6 = 499.6 \) units, when its weight is actually 453.6 units. Since 1 pound = 453.6 grams, the instrument is a balance calibrated in grams and overweighing by 46 grams.

**SHORT ANSWER**

A balance calibrated in grams and overweighing by 46 grams.

---

²Adapted from J.G. Greeno, *Elementary Theoretical Psychology*, (Addison-Wesley, Massachusetts, 1968).
Chapter 1 Section 3

1. The table

<table>
<thead>
<tr>
<th>Height (m)</th>
<th>0</th>
<th>2000</th>
<th>4000</th>
<th>6000</th>
<th>8000</th>
<th>10,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temp (°C)</td>
<td>15</td>
<td>2</td>
<td>−11</td>
<td>−24</td>
<td>−37</td>
<td>−50</td>
</tr>
</tbody>
</table>

shows the air temperature $T$ in degrees Celsius as a function of the height $h$ in meters above the earth’s surface.\(^3\) How well does the equation $T(h) = b + mh$ fit this data set? Complete the following rule of thumb: “For every 1000 meters rise in height, the air temperature...”

**ANSWER**

Since the temperature ($T$) decreases by 13°C for every increase of 2000 meters ($h$), the temperature could be a linear function with slope $= -13/2000 = -0.0065$. If $T$ is a linear function of $h$, we have $T = b + mh$ and $m = -0.0065$. To find $b$, we substitute any point from the table, such as $h = 0$, $T = 15$ into this equation: $15 = b + (-0.0065)(0)$, so $b = 15$. The equation of the line is $T = 15 - 0.0065h$, and this fits the data set perfectly. When $h = 1000$, $T = 15 + (-0.0065)(1000) = 8.5$. Thus, for every 1000 meters rise in height, the air temperature falls by $15 - 8.5 = 6.5°C$.

**SHORT ANSWER**

Perfectly. For every 1000 meters rise in height, the air temperature falls by 6.5°C.

---

1. When deer are confined to a particular area, their birth and death rates depend on the population density of the herd. The following graph shows the annual birth and death rates per 100 deer of a particular herd as a function of the population density in deer per square mile.\(^4\)

![Graph showing birth and death rates per 100 deer vs. population density.]

[Plot \(y = 100 - x\) and \(y = x\) with \(10 \leq x \leq 60\), and \(0 \leq y \leq 100\).]

(a) Which graph represents the annual birth rate, and which graph represents the annual death rate?

(b) If at the beginning of the year 2000 a herd consisted of 1000 deer and they were confined to an area where there were 20 deer per square mile, how many deer were born and how many died by the beginning of 2001? How large was the herd then, and how many deer were there per square mile?

(c) Repeat part (b) for the second year.

(d) Repeat part (c) for the third year.

(e) If at the beginning of the year 2000, instead of a herd consisted of 1000 deer, the herd was large enough so that there were 60 deer per square mile, how large was the herd? How many deer were born and how many died by the beginning of 2001? How large was the herd then, and how many deer were there per square mile?

(f) What do you think the long-term outlook is for herds that are modeled by this situation?

**ANSWER**

(a) As the number of deer per square mile increases, resources will become more scarce, so the death rate should be an increasing function and the birth rate a decreasing function of deer per square mile. The graph that passes through (20, 20) represents the death rate. The graph that passes through (20, 80) represents the birth rate.

(b) When there are 20 deer per square mile, the birth rate is 80% of the current deer population, that is 800 new deer will be born in the next year. When there are 20 deer per square mile, the death rate is 20% of the current deer population, that is 200 old deer will die in that year. The herd will then have $1000 + 800 - 200 = 1600$ deer. The initial 1000 deer were spread out at 20 deer per square mile, so they must occupy $1000/20 = 50$ square miles. The 2001 herd are thus $1600/50 = 32$ deer per square mile.

(c) When there are 32 deer per square mile, the birth rate is about 68% of the current deer population (1600), that is 1088 new deer will be born in the next year. When there are 32 deer per square mile, the death rate is about 32% of the current deer population, that is 512 old deer will die in that year. The herd will then have $1600 + 1088 - 512 = 2176$ deer. The population density is now $2176/50 = 43.52$, deer per square mile.

(d) When there are 43.52 deer per square mile, the birth rate is about 46.48% of the current deer population (2176), that is 1229 new deer will be born in the next year. When there are 43.52 deer per square mile, the death rate is about 43.52% of the current deer population, that is 947 old deer will die in that year. The herd will then have $2176 + 1229 - 947 = 2458$ deer. The population density is now $2458/50 = 49.16$, deer per square mile.

(e) The population in the 50 square mile area is $60 \times 50 = 3000$ deer. When there are 60 deer per square mile, the birth rate is 40% of the current deer population, that is 1200 new deer will be born in the next year. When there are 60 deer per square mile, the death rate is 60% of the current deer population, that is 1800 old deer will die in that year. The herd will then have $3000 + 1200 - 1800 = 2400$ deer, which is $2400/50 = 48$ deer per square mile.

(f) The herd will gradually reach the situation where the birth rate will equal the death rate, and the number of deer will stabilize. At that stage there will be 50 deer per square mile on 50 square miles, namely 2500 deer.

**SHORT ANSWER**

(a) The graph that passes through (20, 80) represents the birth rate. The graph that passes through (20, 20) represents the death rate.

(b) 800, 200, 1600, 32.

(c) 1088, 512, 2176, 43.52.

(d) 1229, 947, 2458, 49.16.

(e) 3000, 1200, 1800, 2400, 48.

(f) Birth rate = death rate. 2500 deer.
1. When deer are confined to a particular area, their birth and death rates depend on the population density of the herd. The following graph shows the annual birth and death rates per 100 deer as a function of the population density in deer per square mile for two different ranges \(A\) (solid lines) and \(B\) (dotted lines).

![Graph showing birth and death rates per 100 deer as a function of population density for ranges A and B.]

\[
\begin{align*}
\text{Plot } y &= -\frac{1}{5}x + 80, \quad y = \frac{1}{3}x + 40 \quad \text{(solid)} \quad \text{with } 10 \leq x \leq 70, \quad \text{and} \quad 0 \leq y \leq 100; \quad \text{and} \quad y &= -\frac{1}{5}x + 40, \quad y = \frac{1}{3}x + 20 \quad \text{(dotted)} \quad \text{with } 10 \leq x \leq 40, \quad \text{and} \quad 0 \leq y \leq 100.
\end{align*}
\]

(a) In the long run, which of the two ranges is able to support the most deer per square mile?

(b) In the long run, which of the two herds will have the greater life expectancy?

**ANSWER**

(a) In the long run, the deer in range \(A\) will have a population density of 60 deer per square mile, and those in range \(B\) will have a population density of 30 deer per square mile. Thus, \(A\) will support more deer per square mile.

(b) In the long run, 60% of the deer in range \(A\) will be replaced each year, whereas only about 30% of the deer in range \(B\) will be replaced each year. Thus, \(B\) will have the older deer, and the greater life expectancy.

**SHORT ANSWER**

(a) \(A\)

(b) \(B\)

---

1. The table

<table>
<thead>
<tr>
<th>Weight on Earth</th>
<th>100</th>
<th>125</th>
<th>150</th>
<th>175</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight on Moon</td>
<td>16.1</td>
<td>20.1</td>
<td>24.1</td>
<td>28.1</td>
</tr>
</tbody>
</table>

shows the weight (in pounds) of 4 people on the Earth and the Moon. Find the line that best fits these points and use this to construct a rule to translate the weight $E$ of an object on the Earth to its weight $M$ on the Moon. What does this rule say about the weight of an object on the Moon that weighs 0 pounds on the Earth? Is that reasonable? Suggest why this happened and what could be done in future to avoid this problem.

**ANSWER**

Since the weight on the moon ($M$) increases by 4 pounds for every increase of 25 pounds on Earth ($E$), the weight could be a linear function with slope $m = 4/25 = 0.16$. If $M$ is a linear function of $E$, we have $M = b + mE$ and $m = 0.16$. To find $b$, we substitute any point from the table, such as $E = 100$, $M = 16.1$ into this equation: $16.1 = b + (0.16)(100)$, so $b = 0.1$. The equation of the line is $M = 0.1 + 0.16E$, and this fits the data set perfectly. When $E = 0$, $M = 0.1$. Thus, a person who weighs nothing on the earth, weighs 0.1 pounds on the moon. One explanation for this is that the people being tested did not wear the same clothes on the earth and the moon.

**SHORT ANSWER**

Not applicable.

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*Adapted from B. Kastner, *Space Mathematics*, (Dale Seymour, Palo Alto, 1987).*
Chapter 1 Section 5

1. The graphs\(^7\) show the life span (in years) of birds and mammals in captivity as a function of their body size (in kg).

\[
\begin{array}{c|c|c|c|c}
\text{Body Size} & 0 & 10 & 20 & 30 & 40 \\
\hline
\text{Life Span - Birds} & 0 & 10 & 20 & 30 & 40 \\
\text{Life Span - Mammals} & 0 & 10 & 20 & 30 & 40 \\
\end{array}
\]

[Plot \( y = 11.8x^{0.20} \) and \( y = 28.3x^{0.19} \).]

(a) Describe what happens to birds and mammals in captivity as their body sizes increase. If a bird and a mammal have the same size, which has the greater life span in captivity?

(b) What would you expect the life span to be for a body size of 0 kg? For each of the body sizes 10, 20, 30, 40 kg, estimate the life span for birds and mammals, and then use these estimates to plot bird life span as a function of mammal life span for each of these body sizes. Use this graph to support the statement that in captivity “birds tend to live more than twice as long as mammals of the same size”.

(c) The formulas corresponding to the graphs are \( L_M = 11.8W^{0.20} \) and \( L_B = 28.3W^{0.19} \), where \( L_M \) and \( L_B \) are the life spans of the mammals and birds, and \( W \) is body size.

i. In general \( W^{0.19} < W^{0.20} \) so why in the original graph is \( L_M < L_B \)?

ii. Eliminate \( W \) from the formulas for \( L_M \) and \( L_B \) to find \( L_B = 28.3 \left( \frac{L_M}{11.8} \right)^{0.95} \). What has this to do with the graph you produced in part (b)?

\(^7\)Adapted from Knut Schmidt-Nielsen, *Scaling: Why is animal size so important?* page 147 (Cambridge University Press, 1985).
iii. Clearly $L_M = L_B$ when $W = 0$. Is there another value of $W$ that makes $L_M = L_B$? What does this mean in terms of the life spans of birds and mammals? Is this realistic?

**ANSWER**

(a) For both birds and mammals the life span increases with body size. Because the graph of the bird life span is above that of the mammal, the bird has the greater life span.

(b) A body size of 0 would mean no life span! From this and the graph we can construct the following table

<table>
<thead>
<tr>
<th>Body size</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mammal life span</td>
<td>0</td>
<td>19</td>
<td>22</td>
<td>23</td>
<td>25</td>
</tr>
<tr>
<td>Bird life span</td>
<td>0</td>
<td>44</td>
<td>50</td>
<td>54</td>
<td>57</td>
</tr>
</tbody>
</table>

and the corresponding graph

This is very close to a line with slope $57/25 = 2.28$. Thus, on average a bird will live over twice as long as a mammal of the same size.

(c) (i) The coefficient 28.3 is larger than the coefficient 11.3.

(ii) $L_M = 11.8W^{0.20}$ gives $W = \left( \frac{L_M}{11.8} \right)^{1/0.20}$ which, when substituted in $L_B = 28.3W^{0.19}$, gives $L_B = 28.3 \left( \frac{L_M}{11.8} \right)^{0.19/0.20}$. This is the function that is plotted in part (b). Also, $0.19/0.20 = 0.95$ is very close to 1, so the graph looks like the line $L_B = \frac{28.3}{11.8}L_M = 2.398L_M$.

(iii) To have $L_M = L_B$ we would need $11.8W^{0.20} = 28.3W^{0.19}$, or $11.8W^{0.01} = 28.3$, so $W = \left( \frac{28.3}{11.8} \right)^{100} = 9.78 \times 10^{37}$ kg. (The largest animal is the blue whale.
which is about $10^5$ kg. The earth weighs about $6 \times 10^{24}$ kg.) If there were an
bird or mammal of this size, its life span would be $11.8 (9.78 \times 10^{37})^{0.20} =
4.6768 \times 10^8$ years. This is unrealistic.

SHORT ANSWER
(a) Life span increases with body size. Bird.
(b) Not appropriate.
(c) (iii) $9.78 \times 10^{37}$ kg. Unrealistic.
Chapter 1 Section 5

1. Birds range in weight from 3-g hummingbirds to 100-kg ostriches. Their eggs range from 0.3 g to 1 kg, and their incubation periods run from 11 to 90 days. Based on a study of many species of birds, the graph shows how the egg mass ($M_{\text{egg}}$ in grams) is related to the body mass ($M_b$ in grams) of the laying bird and the equation $t_{\text{inc}} = 12.03M_{\text{egg}}^{0.217}$ relates the incubation time ($t_{\text{inc}}$ in days) to the egg mass.8 [Plot $y = 0.277x^{0.77}$]

(a) Would you expect the graph to pass through (0,0)?

(b) Comment on the statement: “Birds twice the size lay eggs twice the size”.

(c) Sketch how the incubation period of an egg is related to the body mass of the laying bird. From this, comment on the statement: “Doubling the body size, doubles the incubation period”.

ANSWER

(a) Yes, because a laying bird with no mass would be dead, and unable to lay an egg.

(b) We see that a 100 g bird lays a 10 g egg, whereas a 200 g bird lays a 16 g egg, not a 20 g egg as would be projected by the statement “Birds twice the size lay eggs twice the size”. In fact, because the graph is concave down the heavier the bird the smaller the net gain in egg weight. The statement is false.

(c) We want to find $t_{\text{inc}}$ as a function of $M_b$. We do this by selecting specific values of $M_b$, estimating from the graph the corresponding values of $M_{\text{egg}}$, and then use the formula $t_{\text{inc}} = 12.03M_{\text{egg}}^{0.217}$ to find the corresponding

---

8Adapted from Knut Schmidt-Nielsen, Scaling: Why is animal size so important? page 35 (Cambridge University Press, 1985).
value of \( t_{\text{inc}} \). For example, with \( M_b = 100 \), the graph gives \( M_{\text{egg}} = 10 \), so 
\[ t_{\text{inc}} = 12.03 (10)^{0.217} = 19.83 \text{ days}. \] 
In this way we construct the table

\[
\begin{array}{cccccccccccc}
M_b & 0 & 100 & 200 & 300 & 400 & 500 & 600 & 700 & 800 & 900 & 1000 \\
M_{\text{egg}} & 0 & 10 & 16 & 22 & 28 & 33 & 38 & 43 & 48 & 52 & 56 \\
t_{\text{inc}} & 0 & 19.83 & 21.96 & 23.53 & 24.79 & 25.69 & 26.49 & 27.21 & 27.87 & 28.36 & 28.82 \\
\end{array}
\]

The graph is

[Plot \( y = 9.105x^{0.167} \).]

The incubation time for a 200 g bird is 21.96 days, and that for a 400 g bird is 24.79 days, so the statement “Doubling the body size, doubles the incubation period” is false.

SHORT ANSWER
(a) Yes.
(b) False.
(c) False.
Chapter 1 Section 10

1. A GPS was used to obtain the altitude in feet at various milemarkers on the Mount Lemmon highway in Tucson, Arizona.

<table>
<thead>
<tr>
<th>Milemarker</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Altitude (feet)</td>
<td>2916</td>
<td>4238</td>
<td>5374</td>
<td>6725</td>
<td>7982</td>
<td>8179</td>
</tr>
</tbody>
</table>

(a) Do the milemarkers start at the bottom or top of the mountain? Plot the altitude as a function of the milemarkers. Do you think this graph is a reasonable profile of Mount Lemmon? Explain.

(b) The following graph shows the air pressure in inches of mercury as a function of altitude.

![Graph showing air pressure as a function of altitude]

[Plot \( y = 30e^{-3.23 \times 10^{-5}x} \) with \( 0 \leq x \leq 9,000 \), and \( 22 \leq y \leq 30 \).]

Construct a table and a graph that shows the air pressure at each of the milemarkers. This process generated a new function from old functions. What is this mathematical process called?

ANSWER

(a) Because the altitude increases with milemarker, the milemarkers start at the bottom of the mountain.
[Plot the data from the table given in the question, and join with a cubic spline.]

If the road was straight and the road was parallel to the ridge line, this would be a reasonable profile of Mount Lemmon. However, it is unlikely that either of these conditions are met, so this will not be reasonable profile.

(b) From the pressure versus altitude graph, we can estimate the pressure at each of the milemarkers. For example, at milemarker 0 the altitude is 2916 feet, and from the graph, that corresponds to about 27.2 inches of mercury.

<table>
<thead>
<tr>
<th>Milemarker</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Altitude (feet)</td>
<td>2916</td>
<td>4238</td>
<td>5374</td>
<td>6725</td>
<td>7982</td>
<td>8179</td>
</tr>
<tr>
<td>Air Pressure (inches of mercury)</td>
<td>27.2</td>
<td>26.2</td>
<td>25.3</td>
<td>24.1</td>
<td>23.2</td>
<td>23.0</td>
</tr>
</tbody>
</table>
Plot the data from the table given in the answer, and join with a cubic spline.

The mathematical process is called composition.

SHORT ANSWER

(a) Bottom. No.
(b) Milemarker

<table>
<thead>
<tr>
<th>Milemarker</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air Pressure</td>
<td>27.2</td>
<td>26.2</td>
<td>25.3</td>
<td>24.1</td>
<td>23.2</td>
<td>23.0</td>
</tr>
</tbody>
</table>

Composition.
1. A vertical jump performed by a person, such as a ballet dancer or a basketball player, is governed by the quadratic equation

\[ y(t) = -16t^2 + at. \]

Here \( y \) is the vertical distance measured in feet from the ground to the center of gravity of the jumper, \( t \) is the time in seconds from take-off, and the positive constant \( a \) is the take-off velocity of the jumper.

(a) Where is this quadratic zero? Where does this quadratic have a maximum? Explain the significance of these two properties of the quadratic in terms of a jumper.

(b) If \( T \) is the time that the jumper is in the air, show that \( T = \frac{a}{16} \). If \( H \) the maximum vertical height of the jumper, show that \( H = \frac{a^2}{64} \).

(c) From part (b) show that the time \( T \) that the jumper is in the air during a vertical jump is related to the height \( H \) of the jump by \( H = 4T^2 \). When a ballet dancer performs a vertical jump, the length of time in the air is determined by the music. Consequently, the height \( H \) of the jump is determined by the tempo of the music. Show that if the music calls for vertical jumps every \( \frac{1}{3} \) second, then \( H = \frac{4}{9} \) feet \( \approx 5.3 \) inches for every ballet dancer, independent of height. During vertical jumps, ballet dancers are expected to point their feet. Discuss the disadvantage of being a ballet dancer with big feet doing vertical jumps. What happens to such ballet dancers should they choose to jump high enough to point their feet?\(^9\)

(d) Show that there are two times when the jumper is at \( \frac{3}{4} H \), namely \( \frac{a}{64} \) and \( \frac{3a}{64} \). When slam-dunking, a basketball player appears to hang in the air at the height of his jump. How long does the player spend in the top 25% of the trajectory? What percentage of the total time does the jumper spend in the top 25% of the trajectory? How does this explain why basketball players appear to hang in the air?

**ANSWER**

(a) \( y(t) = -16t^2 + at = t (-16t + a) \) is zero when \( t = 0 \) and when \( t = \frac{a}{16} \). These are the times the jumper takes off and lands.

\[ y(t) = -16t^2 + at = -16 \left( t^2 - \frac{a}{16} t \right) = -16 \left( t - \frac{a}{32} \right)^2 - \left( \frac{a}{32} \right)^2 \]

has a maximum of \( 16 \left( \frac{a}{32} \right)^2 = \frac{a^2}{64} \) when \( t = \frac{a}{32} \). These are the maximum height the

jumper attains and how long it took to reach that height. Notice that the total
time in the air, \( \frac{a}{16} \), is twice the time, \( \frac{a}{32} \), it takes to reach the maximum height.
In other words, it takes as long to go up as it does to come down.

(b) These follow immediately from part (a).

(c) \( 4T^2 = 4 \left( \frac{a}{16} \right)^2 = \frac{a^2}{64} = H \). When \( T = \frac{1}{3} \) then \( H = 4 \left( \frac{1}{3} \right)^2 \) feet, which
is a little more than 5 inches. Thus, when vertical jumps occur every \( \frac{1}{3} \) second,
the dancers jump only 5 inches, but must straighten their toes in that 5 inch space. A person with big feet would not be able to do that, and in fact, pointing
the toes would mean jumping more than 5 inches, and being out of tune.

(d) The times when the player is at \( \frac{3}{4} H \) will occur when \( \frac{3}{4} H = -16t^2 + at \).
Because \( H = \frac{a^2}{64} \), we solve the quadratic \( 16t^2 - at + \frac{a^2}{64} = 0 \), finding \( t = \frac{a}{64} \) and
\( t = \frac{3a}{64} \). The first time is when the player reaches \( \frac{3}{4} H \) on the way up, and the
second on the way down. The total time spent in the top 25% of the trajectory
is \( \frac{3a}{64} - \frac{a}{64} = \frac{a}{32} \). But \( \frac{a}{32} = \frac{1}{2} T \). Thus, the player spends half the jump time in
a quarter of the jump distance, which explains why the player seems to hang in
the air.

SHORT ANSWER
(a) 0, \( a/16 \), \( a/32 \), \( a^2/64 \).
(b) Not appropriate.
(c) Not appropriate.
(d) Not appropriate.