1 Multi-sets and multinomial coefficients

A multinomial coefficient is associated with each (finite) multiset taken from the set of natural numbers. Such a multiset is given by a list \( k_1, \ldots, k_n \), where numbers may be repeated, and where order does not matter.

If the elements of the multiset are \( k_1, k_2, \ldots, k_n \) and \( k_1 + k_2 + k_3 + \cdots + k_n = k \), then the multinomial coefficient is

\[
\binom{k}{k_1, k_2, k_3, \ldots, k_n} = \frac{k!}{\prod_{i=1}^{n} k_i!}.
\]  

(1)

A multiset of natural numbers determines a type vector \( p_1, p_2, p_3, \ldots \). Thus \( p_j \) is the number of times \( j \) occurs in the list \( k_1, \ldots, k_n \). Then \( p_1 + p_2 + p_3 + \cdots = k \). The multinomial coefficient is also given by

\[
\binom{k}{k_1, k_2, k_3, \ldots, k_n} = \frac{k!}{\prod_{j} (j!)^{p_j}}.
\]

(2)

There is a problem when the multiset has 0s in it, since the type vector does not determine the number of zeros. However, suppose that each \( k_i \geq 1 \). Thus the multiset consists only of strictly positive natural numbers. In that case the type vector determines the multiset. Furthermore, it follows that \( p_1 + p_2 + p_3 + \cdots = n \).

A multiset taken from the set of strictly positive natural numbers with sum \( k \) is called a integer partition of \( k \). Each number \( k_i \) in the sum is called a part. We write \( p(k) \) for the number of integer partitions of \( k \) and \( p(k, n) \) for the number of integer partitions of \( k \) into \( n \) parts.

2 Functions and surjective functions

Let \( A \) have \( k \) points and \( B \) have \( n \) points. Let \( f \) be a function from \( A \) to \( B \). Then \( f \) defines a multi-set taken from \( B \). This may be given by listing some elements of \( B \) (repetitions allowed, order irrelevant). Or it may be given by a multi-index \( N \), a function from \( B \) to the natural numbers. The value of \( N \) on
a point \( y \) in \( B \) is the number of points \( x \) with \( f(x) = y \). The total number of such multi-indices is \( \binom{n}{k} \).

The multi-index \( N \) determines a multi-set \( k_1, \ldots, k_n \) taken from the natural numbers. This multi-set consists of the values of the function \( N \). It has less information in it than the multi-index \( N \), since it has lost the information about which elements of \( B \) index the numbers.

The number of functions from \( A \) to \( B \) with given multi-index \( N \) is determined by the values \( k_1, \ldots, k_n \) of the multi-index \( N \). It is given by the multinomial coefficient

\[
\binom{k}{N} = \binom{k}{k_1, k_2, k_3, \ldots, k_n} = \frac{k!}{\prod_{i=1}^{n} k_i!}.
\]

(3)

To prove this, think of \( B \) as being ordered. Then for each function \( f : A \to B \) with the given multi-indices the inverse images form an ordered family of \( n \) subsets of \( A \) with cardinalities \( k_i \). Give in addition an ordering of each subset. The pairs consisting of a suitable function together with orderings of the subsets correspond to orderings of \( A \). This shows that the number of functions times \( k_1! \cdots k_n! \) is equal to \( k! \).

One obvious consequence of this is that

\[
n^k = \sum_{N} \binom{k}{N}.
\]

(4)

The total number of functions is obtained by summing the number of functions corresponding to each multi-index.

Each set partition of \( A \) partitions \( A \) into disjoint non-empty sets. Each set in the set partition is called a block. Say that there are \( n \) blocks. Write \( B(n) \) for the number of set partitions, and \( S(k, n) \) for the number of sets partitions into \( n \) blocks.

Now consider the case of surjective functions from \( A \) to \( B \). This determines a multi-index \( N \) on \( B \) with each value at least one. The number of such multi-indices is \( \binom{n}{k} \). In particular the values \( k_i \geq 1 \), so we have an integer partition of \( k \) into \( n \) parts. Furthermore, the surjective function determines a set partition of \( A \) into \( n \) blocks. The number of set partitions is \( S(k, n) \). The number of surjective functions is thus \( n!S(k, n) \), since to determine the function all one has to do is to give the image of each of the \( n \) blocks.

From this we see that

\[
n!S(k, n) = \sum_{N \geq 1} \binom{k}{N}.
\]

(5)

The total number of surjective functions is obtained by summing the number of functions corresponding to each multi-index that only strictly positive values.

## 3 Set partitions and integer partitions

For each set partition of \( A \) we have a corresponding integer partition which consists of the sizes of the blocks of the set partition. The corresponding type
vector \( p_j \) is just the number of blocks of size \( j \).

The claim is that the number of set partitions of \( A \) with given integer partition is given by the formula

\[
\left( \begin{array}{c} k \\ k_1 \ k_2 \ k_3 \ \cdots \ k_n \end{array} \right) \frac{1}{\prod_j p_j!} = \frac{k!}{\prod_j (j!)^{p_j}} \frac{1}{\prod_j p_j!}.
\]  

(6)

The proof is the following. Consider an \( n \) element set with multi-index \( N \) having values given by the integer partition. Consider a partition of \( A \) into blocks with sizes given by the integer partition. Each block of size \( k_i \) must map into a point in \( B \) with multi-index value \( k_i \). The surjective function from \( A \) to \( B \) with multi-index \( N \) is determined by two kinds of data: the partition and the mappings from the \( p_j \) blocks of given size \( j \) to the \( p_j \) points in \( B \) with multi-index value \( j \). Therefore the number of partitions times \( \prod_j p_j! \) is the number of functions.

As a consequence we get that the number of set partitions of a \( k \) element set \( A \) into \( n \) parts is

\[
\sum_P k! \frac{1}{\prod_j (j!)^{p_j}} \frac{1}{\prod_j p_j!},
\]  

(7)

where the sum is over all type vectors \( p_1, p_2, p_3, \ldots \) with \( p_1 + p_2 + p_3 + \cdots = n \) and \( p_1 1 + p_2 3 + p_3 3 + \cdots = k \), in other words, over all integer partitions of \( k \) into \( n \) parts.

4 Two kinds of multinomial coefficient

There is another, more illuminating way, to get this kind of result. We look at the number of surjective functions \( f : A \to B \) that define a given integer partition \( p_1, p_2, p_3, \ldots \). This will be the number of surjective functions per multi-index on \( B \) times the number of multi-indices on \( B \) per integer partition. We know that the number of surjective functions with multi-index \( N \) is given by a multinomial coefficient. However a type vector is itself a special kind of multi-index, one defined on the strictly positive natural numbers. So the number of multi-indices on \( B \) giving a particular type vector is also given by a multinomial coefficient

\[
\binom{n}{p} = \frac{n!}{\prod_j p_j!}.
\]  

(8)

The result is that the number of surjective functions with given integer partition is the product of two multinomial coefficients

\[
\left( \frac{k}{N} \right) \left( \frac{n}{p} \right) = \frac{k!}{\prod_j (j!)^{p_j}} \frac{n!}{\prod_j p_j!}.
\]  

(9)

In particular, we recover a variant on the previous result:

\[
n!S(k, n) = \sum_P k! \frac{1}{\prod_j (j!)^{p_j}} \frac{n!}{\prod_j p_j!},
\]  

(10)
where the sum is over all type vectors $p_1, p_2, p_3, \ldots$ with $p_1 + p_2 + p_3 + \cdots = n$ and $p_1 1 + p_2 2 + p_3 3 + \cdots = k$, in other words, over all integer partitions of $k$ into $n$ parts. This is a remarkable formula: It writes the number of surjective functions as a sum over integer partitions of terms each of which is a product of two multinomial coefficients.

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