1 Groups and dual groups

Fourier analysis relates functions on a group $G$ to functions on a dual group $\hat{G}$. Here are some examples.

**Real line and dual real line** $G = \mathbb{R}$ and $\hat{G} = \mathbb{R}$. We have the inverse Fourier transform

$$f(x) = \int_{-\infty}^{\infty} e^{ikx} \hat{f}(k) \frac{dk}{2\pi}$$

where the Fourier transform is

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x) dx.$$ (1)

**Circle and dual integers** $G = T_L$ and $\hat{G} = \delta k \mathbb{Z}$. Here $\delta k = \frac{2\pi}{L}$. We have the Fourier series

$$f(x) = \sum_{p=-\infty}^{\infty} e^{ip\delta k x} \hat{f}(p\delta k) \frac{\delta k}{2\pi}$$

where the Fourier coefficients are

$$\hat{f}(p\delta k) = \int_{0}^{L} e^{-ip\delta k x} f(x) dx.$$ (3)

**Finite cyclic group and dual finite cyclic group** $G = \Delta x \mathbb{Z}_N$ and $\hat{G} = \delta k \mathbb{Z}_N$. Here $\delta k = \frac{2\pi}{N\Delta x}$. We have the discrete inverse Fourier transform

$$f(n\Delta x) = \sum_{p=0}^{N-1} e^{ip\delta k n\Delta x} \hat{f}(p\delta k) \frac{\delta k}{2\pi}.$$ (5)

where the discrete Fourier transform is

$$\hat{f}(k) = \sum_{n=0}^{N-1} e^{-ip\delta k n\Delta x} f(n\Delta x) \Delta x.$$ (6)
2 Poisson summation formula

The setting for the Poisson summation formula is a function $f$ on a group $G$ and its Fourier transform $\hat{f}$ on the dual group $\hat{G}$. However now there is a given subgroup $H \subseteq G$. Furthermore, there is a quotient group $G/\!\mod H$. The dual group of $G/\!\mod H$ is a subgroup of the dual group of $G$. The Poisson summation related a sum involving $f$ over $H$ with a sum involving $\hat{f}$ over the dual group of $G/\!\mod H$.

**Integer subgroup of real line** $G = \mathbb{R}$ and $\hat{G} = \mathbb{R}$. The subgroup $H = \Delta x \mathbb{Z}$, and the quotient group $G/\!\mod H = T_{\Delta x}$ with dual group $\Delta k \mathbb{Z}$, where $\Delta k = \frac{2\pi}{\Delta x}$. We have

$$\sum_{n=-\infty}^{\infty} f(x + n\Delta x) = \sum_{m=-\infty}^{\infty} e^{im\Delta k x} \hat{f}(m\Delta k) \frac{\Delta k}{2\pi}$$

(7)

and also

$$\sum_{m=-\infty}^{\infty} \hat{f}(k + m\Delta k) = \sum_{n=-\infty}^{\infty} e^{-in\Delta x} f(n\Delta x) \Delta x.$$  

(8)

The last equation says that if we try to compute the Fourier transform of the function $f(x)$ by a discrete sum with spacing $\Delta x$, then we get the answer $\hat{f}(k)$ plus an infinite number of alias results $\hat{f}(k + m\Delta k)$ with $m \neq 0$.

**Finite cyclic subgroup of circle** $G = T_L$ and $\hat{G} = \delta k \mathbb{Z}$, where $\delta k = \frac{2\pi}{L}$. The subgroup $H = \Delta x \mathbb{Z}_N$, where $N\Delta x = L$. The quotient group $G/\!\mod H = T_{\Delta x}$ with dual group $\Delta k \mathbb{Z}$, where $\Delta k = \frac{2\pi}{\Delta x} = N\delta k$. We have

$$\sum_{n=0}^{N-1} f(x + n\Delta x) = \sum_{m=-\infty}^{\infty} e^{im\Delta k x} \hat{f}(m\Delta k) \frac{\Delta k}{2\pi}$$

(9)

and

$$\sum_{m=-\infty}^{\infty} \hat{f}(p\delta k + m\Delta k) = \sum_{n=0}^{N-1} e^{-ip\delta k n\Delta x} f(n\Delta x) \Delta x.$$  

(10)

The last equation says that if we try to compute the Fourier coefficients of the periodic function $f(x)$ by a finite sum with spacing $\Delta x$, then we get the answer $\hat{f}(p\delta k)$ plus an infinite number of alias results $\hat{f}(p\delta k + m\Delta k)$ with $m \neq 0$. 

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