1. Introduction

Let $\Gamma$ be a connected finite graph; by $V$ we denote the set of its vertices, and by $E$ we denote the set of its edges. If each edge $e$ is considered as a segment of certain length $l(e) > 0$ then such a graph is called a metric graph. One can find a good survey and numerous references in [K]. A metric graph with a given combinatorial structure $\Gamma$ is determined by a vector of edge lengths $(l(e)) \in \mathbb{R}^{|E|}$. We will use the notation $G = (\Gamma, (l(e)))$. The length of a metric graph, $l(G)$, is the sum of the lengths of all its edges. Sometimes, it is convenient to treat each edge as a pair of oriented edges; then, on an oriented edge, one defines a coordinate $x_e$ that runs from 0 to $l(e)$. If $-e$ is the same edge, with the opposite orientation, then $x_{-e} = l(e) - x_e$. If an edge $e$ emanates from a vertex $v$, we express it by writing $v \prec e$.

A function $\phi$ on $G$ is a collection of functions $\phi_e(x)$ defined on each edge $e$. We say that it belongs to $L^2(G)$ if each function $\phi_e$ belongs to $L^2$ on the corresponding edge; then

$$||\phi||^2 = \sum_e ||\phi_e||^2.$$  

The Sobolev space $H^1(G)$ is defined as the space of continuous functions on $G$ that belong to $H^1$ on each edge. The Laplacian on $G$ is defined via the quadratic form

$$\int_G |\phi'(x)|^2 dx = \sum_{e \in E} \int_0^{l(e)} |\phi'_e(x)|^2 dx$$

considered on the natural domain $H^1(G)$. The Laplacian $\Delta$ is given by the differential expression $-d^2/dx^2$ on each edge. Its domain is the set of continuous functions that belong to the Sobolev space $H^2$ on each edge and that satisfy the Kirchhoff condition

$$\sum_{e \prec v} \frac{d\phi}{dx_e} (v) = 0$$  

\hspace{-1cm}(1.1)

The first version of this paper dealt with the smallest positive eigenvalue only; the proof was completely different. Y. Colin de Verdiere and S. Gallot suggested the use of the symmetrization technique. As a result, the theorem became more general and the proof became simpler. My great thanks to them.
for every vertex $v$. This operator is self-adjoint, and its spectrum consists of eigenvalues
\[
0 = \mu_1(G) < \mu_2(G) \leq \mu_3(G) \leq \cdots < \infty
\]
of finite multiplicity. The eigenvalues are the numbers for which the problem
\[
(1.2) \quad \frac{d^2 \phi_e}{dx_e^2} + \lambda \phi_e = 0,
\]
subject to the Kirchhoff conditions (1.1), has a non-trivial solution. For the sake of brevity, we will call $\{\mu_j(G)\}$ the spectrum of the metric graph $G$.

In this paper, we study the extremal properties for $\mu_j(G)$ in the class of metric graphs with a fixed length $\ell$. First, let us make explicit computations for three simple examples.

**Example 1.** $\Gamma$ is a cyclic graph with $k$ vertices $v_1, \ldots, v_k$. It has $k$ edges that connect $v_1$ with $v_2$, $v_2$ with $v_3$, ..., $v_k$ with $v_1$. Obviously, the spectrum of the Laplacian on such a graph is the same as the spectrum of the Laplacian on a circle of circumference $\ell = l(G)$, so
\[
(1.3) \quad \mu_1(G) = 0, \quad \mu_{2k}(G) = \mu_{2k+1}(G) = 4\pi^2 k^2 l(G)^{-2}, \quad k \geq 1.
\]

**Example 2.** $\Gamma$ is a linear graph with $k$ vertices. It is the same graph as in the previous example, with the edge connecting $v_k$ and $v_1$ removed. The spectrum of the Laplacian on such a graph coincides with the spectrum of the Neumann Laplacian on the interval $[0, \ell]$, so
\[
(1.4) \quad \mu_k(G) = \pi^2 (k-1)^2 l(G)^{-2}.
\]

**Example 3.** $\Gamma$ is a star with $k$ edges. It has $k+1$ vertices $v_0, v_1, \ldots, v_k$, and $v_0$ is connected with all other vertices. We assume that $k \geq 2$; in the case when $k = 2$, $\Gamma$ is a linear graph. For a metric graph $G = H_k$, we take the lengths of all edges to be equal to $\ell/k$. Let us orient an edge $e_j$ that connects $v_j$ with $v_0$ toward $v_0$. Then an eigenfunction of the Laplacian on $e_j$ must be of the form $a_j \cos(\sqrt{\lambda} x_j)$ because it satisfies the Neumann condition at $x_j = 0$. If $l\sqrt{\lambda}/k \neq -(\pi/2) + \pi m, m \in \mathbb{Z}_+$, then this function does not vanish at $v_0$, all $a_j$ must be equal to each other, and the Kirchhoff condition (1.1) is satisfied if $\sin(l\sqrt{\lambda}/k) = 0$, or $l\sqrt{\lambda}/k = \pi m, m \in \mathbb{Z}_+$.

One gets a family of simple eigenvalues $\pi^2 k^2 m^2 / l(G)^2$, $m \in \mathbb{Z}_+$, of the Laplacian. If $l\sqrt{\lambda}/k = -(\pi/2) + \pi m$ then the function vanishes at $v_0$, and it is continuous for all values of $a_j$. The Kirchhoff condition at $v_0$ is equivalent to $a_1 + \cdots + a_k = 0$. Therefore,
\[
\lambda = \pi^2 k^2 (2m-1)^2 / 4l(G)^2, \quad m \in \mathbb{Z}_+,
\]
are also eigenvalues of the Laplacian; their multiplicity equal $k-1$. We see that, for a star,
\[
(1.5) \quad \mu_2(H_k) = \mu_k(H_k) = \frac{\pi^2 k^2}{4l(H_k)^2}.
\]

The third example shows that, in the class of metric graphs of fixed length, $\mu_2(G)$, and, therefore, $\mu_j(G)$, $j \geq 2$, does not admit an upper bound. The best
lower bound for $\mu_j(G)$, $j \geq 2$, can be seen when $G = H_j$. The main purpose of this paper is to prove that, in fact, the smallest possible value for $\mu_j(G)$ is achieved when $G = H_j$.

Obviously, one can always remove vertices of degree 2 from the list of vertices. To make some statements simpler, from this point, we assume that there are no vertices of degree 2 in $G$.

**Theorem 1.** Let $G$ be a connected metric graph. Then

$$
\mu_j(G) \geq \frac{\pi^2 j^2}{4l(G)^2}, \quad j \geq 2.
$$

Moreover, an equality in (1.6) occurs if and only if $G$ is a segment when $j = 2$ and $G = H_j$ when $j \geq 3$.

**Remark.** It is known that, in the class of bounded, connected planar domains of given area, $\Omega$, the first eigenvalue $\lambda_1(\Omega)$ of the Dirichlet Laplacian in $\Omega$ is minimized when $\Omega$ is a circle, and the first positive eigenvalue $\mu_2(\Omega)$ of the Neumann Laplacian in $\Omega$ assumes its maximal value when $\Omega$ is a circle [PS]. Moreover $\lambda_1(\Omega)$ can be arbitrarily big, and $\mu_2(\Omega)$ can be arbitrarily close to 0. Though it may look like the eigenvalues of a metric graph should be analogues of the eigenvalues of the Neumann Laplacian: the domain of the Dirichlet functional in the variational formulation is the whole space $H^1(G)$, their extremal properties are closer to those of the eigenvalues of the Dirichlet Laplacian.

2. Proof of Theorem 1

First, it is sufficient to prove the inequality in Theorem 1 for trees. In fact, let $G$ be a metric graph, and let $G'$ be the graph that is obtained from $G$ by cutting an edge $e$ at some point $x_0$. This point gives rise to two different vertices in $G'$. Obviously, $H_1(G) \subset H_1(G')$, so $\mu_j(G) \geq \mu_j(G')$ because $\mu_j(G)$ is obtained by the min-max principle from the Rayleigh quotient over a smaller space. If $G$ is not a tree, one can cut several edges of $G$ to make a connected tree out of it, and the $j$-th eigenvalue of that tree does not exceed $\mu_j(G)$.

Let $G$ be a connected metric tree. By $\phi_1(x) = \text{const}, \phi_2(x), \ldots$, we denote the eigenfunctions of the Laplacian on $G$ that correspond to the eigenvalues $\mu_1 = 0, \mu_2, \ldots$. Fix an integer $j \geq 2$. For any collection of points $x_1, \ldots, x_m \in G$, $m \leq j - 1$, one can find a non-zero linear combination, $\phi(x)$, of $\phi_1(x), \ldots, \phi_j(x)$ that vanishes at all those points. One has

$$
\int_G |\phi'(x)|^2 dx \leq \mu_j(G) \int_G |\phi(x)|^2 dx.
$$

The set $G \setminus \{x_1, \ldots, x_m\}$ consists of a certain number of connected components. By $G(x_1, \ldots, x_m)$ we denote the disjoint union of their closures. Each connected component of $G(x_1, \ldots, x_m)$ is a tree. Let us formulate the first lemma that we need.

**Lemma 2.** Let $G$ be a connected metric tree, and let $j \geq 2$ be an integer. Then there exist points $x_1, \ldots, x_m$, $m \leq j - 1$, such that the length of each connected component of $G(x_1, \ldots, x_m)$ does not exceed $l(G)/j$. 

2A. Proof of (1.6) from Lemma 2. We choose points \(x_1, \ldots, x_m\) from Lemma 2. Then, for at least one of the connected components of \(G(x_1, \ldots, x_m)\) (we call it \(G_1\)) \(\phi(x)\) is not identically 0 on \(G_1\), and

\[
\int_{G_1} |\phi'(x)|^2 \, dx \leq \mu_j(G) \int_{G_1} |\phi(x)|^2 \, dx.
\]

When restricted to \(G_1\), the function \(\phi(x)\) satisfies the Dirichlet boundary condition at one of its leaves. The next lemma gives a lower bound for the ground state of the Laplacian with the Dirichlet condition at a point.

For a metric graph \(G\) and a point \(y \in G\), we denote by \(H_y^1(G)\) the space of \(H^1(G)\) functions that vanish at \(y\).

**Lemma 3.** Let \(G\) be a connected metric graph and \(y \in G\). Then

\[
\int_G |\phi'(x)|^2 \, dx \geq \frac{\pi^2}{4l(G)^2} \int_G |\phi(x)|^2 \, dx
\]

for all functions \(\phi \in H_y^1(G)\). For a non-zero function \(\phi \in H_y^1(G)\), the equality in (2.3) may happen only if \(G\) is a segment, \(y\) is its endpoint, and \(\phi(x)\) is proportional to \(\sin(\pi s/(2l(G)))\) where \(s\) is the distance to \(y\).

One obtains the inequality in Theorem 1 by applying Lemma 3 to \(G_1\) and comparing (2.2) and (2.3).

**Proof of Lemma 3.** We use the symmetrization technique (see [B], [BG], [G1], [G2], [PS].) First, one can assume that \(\phi \geq 0\): replacing \(\phi(x)\) by \(|\phi(x)|\) does not result in the change of either the right hand side or the left hand side in (2.3). For \(t \geq 0\), let \(m_\phi(t)\) be the measure of the set \(\{x \in G : \phi(x) < t\}\); this is a lower semi-continuous function that increases from 0 to \(M = \max \phi(x)\). One can uniquely define a continuous, non-decreasing function \(\phi^*(s)\) on the interval \([0, l(G)]\) such that \(\phi^*(0) = 0\) and \(m_\phi(t) = m_\phi(t)\). Then

\[
\int_G |\phi(x)|^2 \, dx = \int_0^M t^2 \, dm_\phi(t) = \int_0^{l(G)} |\phi^*(s)|^2 \, ds.
\]

The set of \(H_y^1(G)\) functions that are continuously differentiable on closed edges is dense in \(H_y^1(G)\); therefore, for the proof of (2.3), one can assume that \(\phi(x)\) is continuously differentiable on closed edges. A critical point of \(\phi(x)\) is either a critical point on an open edge or a vertex. By Sard’s theorem the set of critical values have measure 0. Let \(t\) be a regular value of \(\phi(x)\). The number of pre-images of \(t\) under \(\phi(x)\) is finite; we denote this number by \(n(t)\). The co-area formula (e.g., see [B]) implies

\[
\int_G |\phi'(x)|^2 \, dx = \int_0^M dt \sum_{x: \phi(x)=t} |\phi'(x)|.
\]

By the Cauchy–Schwarz inequality,

\[
\sum_{x: \phi(x)=t} |\phi'(x)| \geq n(t)^2 \left( \sum_{x: \phi(x)=t} \frac{1}{|\phi'(x)|} \right)^{-1} \geq \left( \sum_{x: \phi(x)=t} \frac{1}{|\phi'(x)|} \right)^{-1} = \frac{1}{m_\phi'(t)}.
\]
Therefore,

\[(2.6) \quad \int_G |\phi'(x)|^2 dx \geq \int_0^M \frac{dt}{m(t)}.\]

The same argument applies to the function $\phi^*(s)$; that function takes every regular value once, and all inequalities become exact equalities. One concludes that

\[\int_G |\phi'(x)|^2 dx \geq \int_0^{l(G)} |(\phi^*)'(s)|^2 ds.\]

Function $\phi^*(s)$ belongs to $H^1([0, l(G)])$ and $\phi^*(0) = 0$. Therefore, \n
\[(2.7) \quad \int_0^{l(G)} |(\phi^*)'(s)|^2 ds \geq \frac{\pi^2}{4l(G)^2} \int_0^{l(G)} |\phi^*(s)|^2 ds\]

because $\pi/(2l(G))$ is the first eigenvalue of the operator $-d^2/ds^2$ on the interval $[0, l(G)]$, with the Dirichlet condition at 0 and the Neumann condition at $l(G)$.

This finishes the proof of the inequality (2.3). Now, suppose that an equality in (2.3) takes place for a non-zero function $\phi(x)$. Then

1. the function $\phi(x)$ minimizes the Rayleigh quotient

   \[\int_G |\phi'(x)|^2 dx / \int_G |\phi(x)|^2 dx\]

   on the space $H^1_0(G)$;

2. the equality in (2.5) holds;

3. the equality in (2.7) holds.

The first condition implies that $\phi(x)$ is an eigenfunction of the Laplacian on $G$, with the Dirichlet condition at the point $y$. Therefore, on each edge of $G \setminus y$, it is a trigonometric function. The same is true for $|\phi(x)|$ because, for that function an equality in (2.3) also holds. The second condition implies that $n(t) = 1$ for all regular values $t$. We conclude that $y$ is a vertex of $G$ of degree 1 (a leaf.) In fact, the derivative of $|\phi(x)|$ at $y$ in each direction emanating from $y$ is positive (it can not vanish), so if there is more than one direction then small positive values are taken at least twice. In the same way, $G$ does not have vertices of degree greater than 2. If $v$ is a vertex of degree at least 3, then, close to $v$, the function $\phi(x)$ either increases or decreases along each edge; so either $\phi(x)$ or $-\phi(x)$ increases in a neighborhood of $v$ along two different edges emanating from $v$. Therefore the values that are close to $\phi(v)$ either from above or from below are taken at least twice.

We have agreed to disregard vertices of degree 2. Finally, $G$ is a connected graph, and all its vertices are leaves. There is at least one vertex ($y$.) Therefore, $G$ is a segment $[0, l(G)]$, and $\phi(x)$ is a monotone function on that segment. That implies $\phi = \phi^*$. The third condition tells us that $\phi^*$ is the first eigenfunction of the Laplacian on $[0, l(G)]$, with the Dirichlet condition at 0 and the Neumann condition at $l(G)$, so it is proportional to $\sin(\pi s/(2l(G)))$. □

2B. Proof of Lemma 2. The proof of Lemma 2 is based on the following lemma.
Lemma 4. Let $G$ be a connected metric tree of length $L$. For every $l$, $0 < l \leq L$, there exists a point $x \in G$ such that

$$G(x) = G_0 \sqcup G_1 \sqcup \cdots \sqcup G_p,$$

and $l(G_0) \leq L - l$, $l(G_k) \leq l$, $1 \leq k \leq p$.

One applies Lemma 4 $(j - 1)$ times. Fix $l = L/j$. First, one finds a point $x_1$ such that $G(x_1) = G_1 \sqcup G^{(1)}$, where $G_1$ is a connected tree of length $\leq (j - 1)L/j$, and all connected components of $G^{(1)}$ have length $\leq L/j$. Then one finds $x_2 \in G_1$ such that $G_1 = G_2 \sqcup G^{(2)}$, with $G_2$ being a connected tree of length $\leq (j - 2)L/j$, and all connected components of $G^{(2)}$ having length $\leq L/j$. One keeps going, and, after not more than $(j - 1)$ steps, one gets the desired decomposition.

Proof of Lemma 4. We fix a leaf $y_0$ of $G$. For a point $x \in G$ that is not a vertex, we denote by $G_x$ the connected component of $G(x)$ that does not contain $y_0$. Note that, if $x$ is not a vertex, then $G(x)$ consists of exactly two connected components. If $l(G_x) = l$ for some $x \in G \setminus V$ (here $V$ is the set of vertices) then such a point will do the job. Otherwise, on each edge $e$ of $G$, either $l(G_e) < l$, $x \in e$, (we call them edges of the first type) or $l(G_e) > l$, $x \in e$; they will be called edges of the second type. Denote by $G^1$ the closure of the union of all edges of the first type; $G^2$ is the closure of the union of edges of the second type. All connected components of both $G^1$ and $G^2$ are metric trees. Notice that the edge incident to $y_0$ belongs to $G^2$, and the edges that are incident to all other leaves of $G$ belong to $G^1$. Let $y \neq y_0$ be a leaf of $G_2$. By $G_0$ we denote the component of $G(y)$ that contains $y_0$, and let $G_1, \ldots, G_p$ be other components of $G(y)$.

We claim that $l(G_0) \leq L - l$ and $l(G_k) \leq l$, $1 \leq k \leq p$. In fact, let $e_k$, $0 \leq k \leq p$, be the edge of $G_k$ incident to $y$ (notice that $y$ is a leaf for all $G_j$s.) For $x \in e_0$, one has $l(G_x) \geq l$, and

$$l(G_0) = \lim_{e_0 \ni x \to y} l(G \setminus G_x) \leq L - l.$$

Because $y$ is a leaf of $G^2$, the edges $e_1, \ldots, e_p$ belong to $G^1$; therefore, for $1 \leq k \leq p$, one has

$$l(G_k) = \lim_{e_k \ni x \to y} l(G_x) \leq l.$$

2C. The case of equality in (1.6). To finish the proof of Theorem 1 we have to analyze, under what conditions the equality in (1.6) takes place. First, we consider the case when $G$ is a connected tree. Then, for any linear combination $\phi(x)$ of $\phi_1(x), \ldots, \phi_j(x)$ that vanishes at the points $x_1, \ldots, x_m$ from Lemma 2, the inequality (2.1) becomes an exact equality. Therefore, $\phi(x)$ is an eigenfunction of the Laplacian on $G$ that corresponds to the eigenvalue $\mu_j(G) = \pi^2 j^2/(4l(G)^2)$. Let $G_1, \ldots, G_p$ be the connected components of $G(x_1, \ldots, x_m)$. The restriction of $\phi(x)$ to $G_k$, $k = 1, \ldots, p$, if not identically zero, is an eigenfunction of the Laplacian on $G_k$ with the Dirichlet condition at those points $x_i$ that belong to $G_k$. From Lemma 3 (notice that the length of each $G_k$ does not exceed $l(G)/j$) we conclude that those components $G_k$, on which the function $\phi(x)$ does not vanish identically, are segments of length $l(G)/j$, one endpoint of each segment is one of the points $x_1, \ldots, x_m$, and the restriction of $\phi(x)$ to such a segment is proportional to $\sin(\pi js/(2l(G)))$ where $s$ is the distance to the endpoint of the segment where
\(\phi(x)\) vanishes. The function \(\phi(x)\) does not vanish at the second end of the segment, so the second end of the segment is a leaf of the tree \(G\) because this segment is a connected component of \(G(x_1, \ldots, x_m)\).

A certain complication arises from the fact that \(\phi(x)\) may vanish on some of the components \(G_j\): an eigenfunction of the Laplacian on a metric graph may well vanish on some edges of the graph. Now, we do induction in \(j\). If \(j = 2\) then \(m = 1\), and one has only one point \(x_1\). The function \(\phi(x)\) does not vanish on at least two connected components of \(G(x_1)\): otherwise \(\phi(x)\) would not satisfy the Kirchhoff condition at the point \(x_1\) (notice that \(x_1\) is not a leaf of \(G\); if \(x_1\) is not a vertex then the Kirchhoff condition is the same as the differentiability at \(x_1\) condition.) Each connected component of \(G(x_1)\) on which \(\phi(x)\) does not vanish is of length \(l(G)/2\), so there are exactly two of them, and these are the only connected components of \(G(x_1)\). We conclude that \(G\) consists of two segments of length \(l(G)/2\) emanating from \(x_1\), so \(G\) is a segment, and \(x_1\) is its midpoint.

Now, let us do the inductive step. Let \(j \geq 3\). Let \(G_1\) be a connected component of \(G(x_1, \ldots, x_m)\) on which \(\phi(x)\) does not vanish. Suppose that \(x_1\) is an endpoint of \(G_1\). As we have already seen, \(G_1\) is a segment of length \(l(G)/j\) than connects \(x_1\) with a leaf of the graph \(G\). Therefore, \(G' = G\setminus G_1\) is a connected tree, \(x_1\) is one of its vertices, and \(l(G') = (j - 1)l(G)/j\). By \(L\) we denote the space of all linear combinations of \(\phi_1(x), \ldots, \phi_j(x)\) that vanish at \(x_1\). Clearly, \(\dim L = j - 1\). A non-zero function \(\psi(x) \in L\) can not vanish identically on \(G'\). In fact, if it vanishes on \(G'\), then

\[
\int_{G_1} |\psi'|^2 dx \leq \frac{\pi^2}{4l(G_1)^2} \int_{G_1} |\psi(x)|^2 dx,
\]

so the restriction of \(\psi(x)\) to \(G_1\) is proportional to \(\sin(\pi s/(2l(G_1)))\), and the Kirchhoff condition breaks at the point \(x_1\). Denote by \(L'\) the space of restrictions of functions from \(L\) to \(G'\). Then

\[
\dim L' = j - 1.
\]

For every \(\psi \in L\), one has

\[
\int_{G} |\psi'(x)|^2 dx \leq \frac{\pi^2 j^2}{4l(G)^2} \int_{G} |\psi(x)|^2 dx
\]

and

\[
\int_{G_1} |\psi'(x)|^2 dx \geq \frac{\pi^2 j^2}{4l(G)^2} \int_{G_1} |\psi(x)|^2 dx.
\]

Therefore,

\[
\int_{G'} |\psi'(x)|^2 dx \leq \frac{\pi^2 j^2}{4l(G')^2} \int_{G'} |\psi(x)|^2 dx = \frac{\pi^2 (j - 1)^2}{4l(G')^2} \int_{G'} |\psi(x)|^2 dx.
\]

From (2.8) and (2.9), one concludes that

\[
\mu_{j-1}(G') \leq \frac{\pi^2 (j - 1)^2}{4l(G')^2},
\]
and, by the induction assumption, $G' = H_{j-1}$. In the case $j = 3$, we treat a segment as $H_2$ by inserting a vertex at the midpoint of the segment. Denote by $y$ the center of $G' = H_{j-1}$. The question is, how the segment $G_1$ is attached to $G'$. There are three possibilities:

1. $x_1 = y$;
2. $x_1$ lies inside of an edge $e = (y, z)$ of $G'$;
3. $x_1$ coincides with a leaf $z$ of $G'$.

In the first case, $G = H_j$, so we have to rule out two remaining possibilities.

Suppose that $x_1$ lies inside of $(y, z)$. Let $G'' = G' \setminus (x_1, z]$. Every function $\psi \in L'$ satisfies

\[
\int_{G''} |\psi'(x)|^2 dx \leq \frac{\pi^2(j - 1)^2}{4l(G')^2} \int_{G'} |\psi(x)|^2 dx
\]

because

\[
\int_{(x_1, z)} |\psi'(x)|^2 dx \geq \frac{\pi^2(j - 1)^2}{4l(G')^2} \int_{(x_1, z)} |\psi(x)|^2 dx
\]

(notice that the length of $(x_1, z)$ is smaller than $l(G)/j = l(G')/(j-1)$.) A function $\psi \in L'$ can not vanish on $G''$ because, otherwise, a strict inequality would hold in (2.11), and that would contradict (2.9). Therefore, the inequality (2.10) holds for functions from a $(j - 1)$-dimensional subspace of $H^1(G'')$. Hence,

\[
\mu_j-1(G'') \leq \frac{\pi^2(j - 1)^2}{4l(G')^2} < \frac{\pi^2(j - 1)^2}{4l(G''')^2}.
\]

The last inequality contradicts (1.6).

Let us now treat the case $x_1 = z$. Then the graph $G$ consists of $(j - 2)$ edges, $e_1, \ldots, e_{j-2}$ emanating from $y$, of length $L/j$ each, and one edge, $f$, of length $2L/j$ emanating from $y$ (here $L = l(G)$.) All edges connect $y$ with leaves of $G$. We parametrize each edge by the distance from $y$. An eigenfunction of the Laplacian on $G$ that corresponds to an eigenvalue $\mu = \lambda^2 \neq 0$ equals $a_k \cos(\lambda((L/j) - s))$ on an edge $e_k$, and it equals $b \cos(\lambda((2L/j) - s))$ on the edge $f$. When $s = 0$, all the values must coincide, so

\[
\begin{align*}
 a_1 \cos(\lambda L/j) = \cdots = a_{j-2} \cos(\lambda L/j) = b \cos(2\lambda L/j).
\end{align*}
\]

The Kirchhoff condition at $y$ reads

\[
(\sum_{k=1}^{j-1} a_k \sin(\lambda L/j)) + b \sin(2\lambda L/j) = 0.
\]

We will count the number of eigenvalues of the Laplacian on $G$ that do not exceed $\pi^2 j^2/(4L^2)$. There is an eigenvalue 0 of multiplicity 1. In the case $\cos(\lambda L/j) = 0$, (2.12) and (2.13) imply $a_1 + \cdots + a_{j-2} = 0$ and $b = 0$; one gets a $(j - 3)$-dimensional space of eigenfunctions that correspond to the eigenvalue $\pi^2 j^2/(4L^2)$. If $\sin(\lambda L/j) = 0$ then $\mu = \lambda^2 \geq (\pi^2 j^2/L^2) > \pi^2 j^2/(4L^2)$. In the case when $\cos(\lambda L/j) \neq 0$ and $\sin(\lambda L/j) \neq 0$, (2.12) and (2.13) imply $a_1 = \cdots = a_{j-2} = a$,

\[
 a \cos(\lambda L/j) = b(2 \cos^2(\lambda L/j) - 1), \quad \text{and} \quad \cos^2(\lambda L/j) = \frac{j - 2}{2(j - 1)}.
\]
Therefore, \( \cos(2L\lambda/j) = -1/(j - 1) \). This case gives rise to one eigenvalue \( \arccos(-1/(j - 1))j^2/(4L^2) \) of multiplicity one that is smaller than \( \pi^2j^2/(4L^2) \); all other eigenvalues are bigger than \( \pi^2j^2/(4L^2) \). Finally, in the case \( x_1 = z \), there are exactly \( (j - 1) \) eigenvalues of \( G \) that are smaller than or equal to \( \pi^2j^2/(4\mu(G)^2) \), so, for such a graph, an equality in (1.6) does not take place.

We have proved that if an equality in (1.6) takes place, and if \( G \) is a connected tree, then \( G = H_j \). If \( G \) is a connected graph that is not a tree then one can cut it at points \( x_1, \ldots, x_m \) lying on open edges in such a way that \( G' = G(x_1, \ldots, x_m) \) is a connected tree. As it was noted earlier, \( \mu_j(G') \leq \mu_j(G) \). If \( \mu_j(G) = \pi^2j^2/(4\mu(G)^2) \) then the last inequality, in combination with (1.6), imply \( \mu_j(G') = \pi^2j^2/(4\mu(G')^2) \). Therefore, \( G' = H_j \) for any choice of points \( x_1, \ldots, x_m \) that make \( G(x_1, \ldots, x_m) \) a connected tree. Clearly, this is impossible. □

References


