Propagation of singularities for the wave equation

Let \( u(t, x) \) be the solution of the initial value problem for the wave equation in \( \mathbb{R}^n \):

\[
\begin{align*}
  u_{tt} &= \Delta u, \\
  u(0, x) &= 0, \\
  u_t(0, x) &= f(x).
\end{align*}
\]  

(1)

At this point, I assume that \( f(x) \in L^2(\mathbb{R}^n) \) and that the function \( f(x) \) has compact support. In the end of these notes, I discuss the general case \( f \in \mathcal{D}'(\mathbb{R}^n) \). To solve the problem (1), one makes the Fourier transform \( u(t, x) \mapsto \hat{u}(t, \xi), f(x) \mapsto \hat{f}(\xi) \). Then

\[
\hat{u}_{tt} + |\xi|^2 \hat{u} = 0, \quad \hat{u}(0, \xi) = 0, \quad \hat{u}_t(0, \xi) = \hat{f}(\xi).
\]

The last problem can be easily solved:

\[
\hat{u}(t, \xi) = \frac{\sin(t|\xi|)}{|\xi|} \hat{f}(\xi).
\]  

(2)

Notice that formula (2) makes perfect sense when \( f \) is a distribution; then \( \hat{f} \) is a distribution, and \( \hat{u}(t) \) is a distribution for any fixed value of \( t \). Our goal is to localize the wave front set of \( u(t) \) in terms of the wave front set of \( f \). By \( u(t) \) I denote the function \( u(t, \cdot) \).

The formula (2) implies immediately that \( \hat{u}(t, \xi) \) is rapidly decaying (by “rapidly decaying” I always mean “decaying faster than any negative power of \( 1 + |\xi| \)” exactly in the directions where \( \hat{f}(\xi) \) is rapidly decaying. Therefore

\[
\Sigma(u(t)) = \Sigma(f).
\]  

(3)

In particular, the solution is a smooth function if the initial condition is smooth. We take the inverse Fourier transform of \( \hat{u}(t, \xi) \) to get

\[
u(t, x) = (2\pi)^{-n} \int \frac{\sin(t|\xi|)}{|\xi|} e^{ix\xi} \hat{f}(\xi) d\xi.
\]  

(4)

It is convenient to express the sin function as a combination of exponential functions and to break the integral (4) into the sum of two integrals. However, if one does this then each integral will be divergent in a neighborhood of \( \xi = 0 \). To avoid this problem, I introduce a cut-off function \( \chi(\xi) \). This is a smooth function that equals 1 when \( |\xi| \geq 1 \), and it vanishes when \( |\xi| \leq 1/2 \). Then

\[
u(t, x) = \left( \frac{2\pi}{2i} \right)^{-n} (u_+(t, x) - u_-(t, x)) + (2\pi)^{-n} \int (1 - \chi(\xi)) \frac{\sin(t|\xi|)}{|\xi|} e^{ix\xi} \hat{f}(\xi) d\xi
\]  

(5)

where

\[
u_\pm(t, x) = \int \frac{\chi(\xi)}{|\xi|} e^{i[(x-y)\pm t|\xi|]} \hat{f}(\xi) d\xi.
\]  

(6)
The function $1 - \chi(\xi)$ vanishes outside the unit ball, so the last term in (5) represents a smooth function. Therefore, $WF(u(t)) \subset WF(u_+(t)) \cup WF(u_-(t))$. Notice that $u_-(t) = u_+(t)$, so it is sufficient to study $WF(u_+(t))$; then $WF(u_-(t)) = WF(u_+(t))$.

First, we prove the following lemma.

**Lemma 1.** Let $\Sigma_1(f) = \Sigma(f) \cap \{\xi : |\xi| = 1\}$. Then

$$
sing\ supp(u_+(t)) \subset supp(f) - t\Sigma_1(f).
$$

For two sets $A, B \subseteq \mathbb{R}^n$, by $A - B$ we denote the set of all differences $x - \xi$ where $x \in A$ and $\xi \in B$.

**Proof.** Let $x_0$ be a point that does not belong to $supp(f) - t\Sigma_1(f)$. We have to show that $x_0 \notin sing\ supp(u_+(t))$. Choose a neighborhood $U$ of the point $x_0$ and a conic neighborhood $\Gamma$ of $\Sigma(f)$ such that

$$
U \cap (supp(f) - t\Gamma) = \emptyset;
$$

here $\Gamma = \Gamma \cap \{\xi : |\xi| = 1\}$. We introduce a cut-off function $\psi(\xi)$ that satisfies the following properties:

(i) $\psi(\xi) \in C^\infty(\mathbb{R}^n \setminus \{0\})$;

(ii) $\psi(\xi)$ is homogeneous of degree 0, i.e. $\psi(t\xi) = \psi(\xi)$ for every $t > 0$;

(iii) $\psi(\xi) = 1$ when $\xi$ belongs to a neighborhood of $\Sigma(f)$;

(iv) $\psi(\xi) = 0$ when $\xi \notin \Gamma$.

We break the integral that represents $u_+(t, x)$ (see (6)) into the sum $u_1^+(t, x) + u_2^+(t, x)$ where

$$
u_1^+(t, x) = \int \frac{\psi(\xi)\chi(\xi)}{|\xi|} e^{i(x\xi + t|\xi|)} \hat{f}(\xi) d\xi
$$

and

$$
u_2^+(t, x) = \int (1 - \psi(\xi))\chi(\xi) e^{i(x\xi + t|\xi|)} \hat{f}(\xi) d\xi.
$$

The Fourier transform $\hat{f}(\xi)$ of the function $f$ is a rapidly decaying function on the support of $1 - \psi(\xi)$, so $u_2^+(t, x)$ is a smooth function, and the singular support of $u_+(t, x)$ is the same as of $u_1^+(t, x)$. Now we use the Fourier transform formula for $\hat{f}(\xi)$ to rewrite (9) in the form

$$
u_1^+(t, x) = \int \int \frac{\psi(\xi)\chi(\xi)}{|\xi|} e^{i(x - y)\xi + t|\xi|} f(y) dy d\xi.
$$

We will do a number of partial integrations in (10). Let $D_j = D_{\xi_j} = (1/i)\partial/\partial \xi_j$. One has

$$
D_j e^{i(x - y)\xi + t|\xi|} = \left(\frac{x - y + t \xi}{|\xi|}\right)_j e^{i(x - y)\xi + t|\xi|}.
$$

Introduce a first order differential operator

$$
L = \sum_{j=1}^n \frac{(x - y + t(\xi/|\xi|))}{|x - y + t(\xi/|\xi|)|^2} D_j.
$$
Notice that the denominators in (12) do not vanish when \( x \in U, \ y \in \text{supp}(f) \) and \( \xi \in \text{supp}(\psi) \) (see (8).) I denote the \( D_j \)-coefficient in (12) by \( a_j(x, y, \xi) \). It is a homogeneous of degree 0 in \( \xi \) function that depends on \( x \) and \( y \) smoothly. The formula (11) implies

\[
Le^{[(x-y)\xi+t|\xi|]} = e^{[(x-y)\xi+t|\xi|]}.
\]

Therefore,

\[
u^1_+(t, x) = \int \int \frac{\psi(\xi) \chi(\xi)}{|\xi|} L^k \left(e^{i[(x-y)\xi+t|\xi|]}f(y)dyd\xi \right)
= \int \int (L^t)^k \left(\frac{\psi(\xi) \chi(\xi)}{|\xi|}\right)e^{i[(x-y)\xi+t|\xi|]}f(y)dyd\xi
\]

where

\[
L^t = -\sum_{j=1}^n a_j(x, y, \xi) D_j + b(x, y, \xi)
\]

with

\[
b(x, y, \xi) = -\frac{1}{i} \sum_{j=1}^n \frac{\partial a_j(x, y, \xi)}{\partial \xi_j}.
\]

The number \( k \) in (13) is arbitrary.

**Reminder.** Let \( z \) be any complex number. A function \( h(\xi) \) defined on \( \mathbb{R}^n \setminus \{0\} \) is called homogeneous of degree \( z \) if

\[
h(\tau \xi) = \tau^z h(\xi), \quad \tau > 0.
\]

A homogeneous function of degree \( z \) is completely determined by its values on the unit sphere in \( \mathbb{R}^n \). We will use the following fact: a partial derivative of a homogeneous function of degree \( z \) is a homogeneous function of degree \( z - 1 \). To see that, one differentiates both sides of (14) with respect to \( \xi_j \). We say that \( h(\xi) \) is homogeneous of degree \( z \) when \( |\xi| \geq R \) is (14) holds for when \( |\xi| \geq R \) and \( \tau \geq 1 \).

As we have already noticed, the coefficients \( a_j \) are homogeneous in \( \xi \) of degree 0. The free term \( b(x, y, \xi) \) in (13) is \( \xi \)-homogeneous of degree \(-1\).

**Exercise.** Show that

\[
(L^t)^k = \sum_{|\alpha| \leq k} a_{\alpha}(x, y, \xi) D^\alpha
\]

where the coefficient \( a_{\alpha}(x, y, \xi) \) is \( \xi \)-homogeneous of degree \( |\alpha| - k \).

In (13), the operator \((L^t)^k\) is applied to the function \( \chi(\xi)\psi(\xi)/|\xi| \), which is homogeneous of degree \(-1\) when \( |\xi| \geq 1 \). The function \( D^\alpha(\chi(\xi)\psi(\xi)/|\xi|) \) is homogeneous of degree \(-1 - |\alpha|\) when \( |\xi| \geq 1 \), so \((L^t)^k(\chi(\xi)\psi(\xi)/|\xi|)\) is homogeneous of degree \(-k - 1\) when
$|\xi| \geq 1$. Differentiating this function with respect to $x$ and $y$ does not change its degree of homogeneity in $\xi$. Therefore, for any multi-index $\beta$,

$$|D_\xi^\beta((L^1)^k(x(\xi)\psi(\xi)/|\xi|))| \leq C_\beta(1 + |\xi|)^{-k-1}$$

(15)

when $x \in U$, $y \in \text{supp}(f)$, and $\xi \in \text{supp}(\psi)$. The function $\hat{f}(\xi)$ is bounded as the Fourier transform of an $L^1$-function, so, by taking $k = n + N$, we make sure that the integral (13) can be differentiated up to $N$ times with respect to $x$. The number $N$ is arbitrary. Therefore, $u_+^1(x) \in C^\infty(U)$. In particular, $x_0 \notin \text{sing supp}(u(t))$.

Q.E.D.

**Corollary 2.**

$$\text{sing supp}(u_+(t)) \subset \text{sing supp}(f) - t\Sigma_1(f).$$

(16)

**Proof.** Suppose that $x_0 \notin \text{sing supp}(f) - t\Sigma_1(f)$. Then there exists a neighborhood $U$ of sing supp$(f)$ such that $x_0 \notin U - t\Sigma_1(f)$. Let $\phi(x) \in C_0^\infty(U)$ and $\phi(x) = 1$ in a neighborhood of sing supp$(f)$. The function $u_+(t)$ is the sum of $u_+^1(t)$ and $u_+^2(t)$ where $u_+^1(t)$ corresponds to the initial condition $\phi(x)f(x)$ and $u_+^2(t)$ corresponds to the initial condition $(1 - \phi(x))f(x)$. Notice that $(1 - \phi(x))f(x) \in C_0^\infty$, so $u_+^2(t) \in C^\infty$. Then, $x_0 \notin \text{supp}(\phi f) - t\Sigma_1(\phi f)$ because $\sigma(\phi f) \subset \Sigma(f)$. By Lemma 1, $x_0 \notin \text{sing supp}u_+(t)$.

Q.E.D.

Now, we are ready to formulate the “propagation of singularities theorem.”

**Theorem.** Let $u(t)$ be the solution of the problem (1) at the moment $t$. Then

$$\text{sing supp}(u(t)) \subset S_+(t) \cup S_-(t)$$

(17)

where

$$S_\pm(t) = \bigcup_{(x,\xi) \in WF(f)}(x \mp t(\xi/|\xi|)).$$

(18)

The Theorem has a simple interpretation. For a point $x_0 \in \text{sing supp}(f)$, draw all the rays that emanate from $x_0$ and that have $\xi$ as its direction vector where $(x_0, \pm \xi) \in WF(f)$. Let $S_{x_0}(t)$ be the union of points that lie on these rays and that are at the distance $t$ from $x_0$. Then

$$\text{sing supp}(u(t)) \subset \bigcup_{x_0 \in \text{sing supp}(f)}S_{x_0}(t).$$

In short, this means that the singularities of the wave equation are propagated with the velocity $\pm 1$ in the directions of the wave front set of the initial data. One can prove a sharper version of the Theorem that gives a localization for the wave front set of $u(t)$, not just for the sing succu$(t)$.

**Proof of the Theorem.** First, I assume that $f(x)$ is an $L^2$ function with compact support. Suppose that $x_0 \notin S_+(t)$. Then $x \notin y - t(\Sigma_0(f) \cap S^{n-1})$ for every $y \in \text{supp}(f)$.

Here, as usual, $S^{n-1}$ is the unit sphere. By Proposition 3 from the notes on the wave front set, there exists a neighborhood $U_y$ of the point $y$ such that

$$x_0 \notin U_y - t\Sigma_1(\phi f)$$

(19)

for every function $\phi \in C_0^\infty(U_y)$. The neighborhoods $U_y$ cover $\text{supp}(f)$. We have assumed that $\text{supp}(f)$ is compact, so one can find a finite number of them, $U_1, \ldots, U_p$, that still
Let \( \{ \phi_j \} \) be a partition of unity that corresponds to the covering \( \{ U_j \} \): functions \( \phi_j(x) \in C_0^\infty(U_j) \) and their sum equals 1 in a neighborhood of \( \text{supp}(f) \). Then the solution of the problem (1) is the sum of the functions \( u_j(t) \) that solve that problem with \( f(x) \) replaced by \( \phi_j(x)f(x) \). Let \( u_{j,\pm} \) be the corresponding “half-solutions”. By Lemma 1, (19) implies that \( x_0 \notin \text{sing supp}(u_{j,+}(t)) \) for all \( j \). Therefore, \( x_0 \notin \text{sing supp}(u_+(t)) \). By replacing \( t \mapsto -t \), one concludes that \( x_0 \notin S_-(t) \) implies \( x_0 \notin \text{sing supp}(u_-(t)) \). This proves the Theorem in the case when \( f \) is an \( L^2 \) function with compact support.

One can easily remove the assumption of \( f \) having a compact support. To do that, one recalls from the standard PDE course that solutions of the wave equation are propagated with speed 1, that is
\[
\text{supp}(u(t)) \subset \{ x : \text{dist}(x, \text{supp} f) \leq |t| \}. \tag{20}
\]
Suppose that the support of \( f \) is not compact. Let \( x_0 \) be a point. We would like to find out whether \( x_0 \in \text{sing supp}(u(t)) \). Take a function \( \psi(x) \in C_0^\infty(\mathbb{R}^n) \) such that \( \psi(x) = 1 \) when \( |x - x_0| \leq |t| + 1 \). Let \( \tilde{u}(t) \) be the solution of the problem (1), with \( f(x) \) replaced by \( \psi(x)f(x) \). The relation (20) implies \( u(x,t) = \tilde{u}(x,t) \) when \( |x - x_0| < 1 \). One notices that \( \text{WF}(\psi f) \subset \text{WF}(f) \) and applies the Theorem to \( \tilde{u}(t) \).

Let \( f \) be a distribution with compact support. Then one can represent \( f \) in the form \( f = (1 - \Delta)^k g \) where \( g(x) \in L^2(\mathbb{R}^n) \). In fact, the Fourier transform of the function \( f \) satisfies the estimate
\[
|\hat{f}(\xi)| \leq C(1 + |\xi|)^m
\]
for some number \( m \). We take
\[
g(x) = (2\pi)^{-n} \int e^{ix\xi} \frac{\hat{f}(\xi)}{(1 + |\xi|^2)^k} d\xi \tag{21}
\]
where \( k \) is chosen in such a way that \( 2k - m > n/2 \) (that guarantees \( \hat{f}(\xi)/(1 + |\xi|^2)^k \in L^2 \)). We will prove later that \( \text{WF}(g) \subset \text{WF}(f) \) (the function \( g \) is a pseudodifferential operator applied to \( f \), and, as we will see, this is a general property of pseudodifferential operators.) Let \( v(t,x) \) be the solution of the problem (1), with \( f \) replaced by \( g(x) \). Then \( u(t) = (1 - \Delta)^k u(t) \), so \( \text{supp} u(t) \subset \text{sing supp} v(t) \). Now, we can apply the theorem to \( v(t) \) and derive the statement about \( u(t) \) from that.

Q.E.D.