Problems for “Introduction to micro-local Analysis”

1. In the space of compactly supported smooth functions in $\mathbb{R}^n$, the base of open neighborhoods of a function $\psi(x)$ is defined as a collection of sets

$$ U_{\gamma(x),\alpha}(\psi) = \{ \phi(x) \in C_0^\infty(\mathbb{R}^n) : |D^\alpha(\phi(x) - \psi(x))| < \gamma(x) \} $$

for all continuous functions $\gamma(x) > 0$ and all multi-indices $\alpha$. Show that the resulting topology induces the following notion of convergence: $\phi_n \rightarrow \phi$ if all functions $\phi_n$ are supported in the same bounded region and $D^\alpha \phi_n$ converges to $D^\alpha \phi$ uniformly for every multi-index $\alpha$.

2. Let $\phi(x) \in C_0^\infty(\mathbb{R}^n)$, and let $s$ be a non-negative real number. Investigate the asymptotic behavior of the integral

$$ I_\phi(\epsilon) = \int_{|x| \geq \epsilon} \frac{\phi(x)}{|x|^{n+s}} dx $$

as $\epsilon \rightarrow 0$. Use the asymptotics of $I_\phi(\epsilon)$ for regularizing the function $|x|^{-n-s}$. The answer depends on whether the number $s$ is integer or not.

3. Prove that the Fourier transform is a continuous map from the Schwarz space $S(\mathbb{R}^n)$ onto itself.

4. Let $u \in S'(\mathbb{R}^n)$ and $\phi(x) \in S(\mathbb{R}^n)$. Prove that the Fourier transform of $\phi \ast u$ equals the product of Fourier transforms of $\phi$ and $u$.

5. Find Fourier transforms of the following distributions:
   a) $P.V(1/x)$;
   b) $(x \pm i0)^{-1}$;
   c) $x_+^\lambda$.

6. Let $\Omega$ be a domain in $\mathbb{R}^n$ with smooth boundary. Find the wave front set of the characteristic function $\chi_\Omega$ of the domain $\Omega$. Do the same problem for

$$ \Omega = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_j \geq 0, j = 1, \ldots, n \}.$$
