Problem 1.4.3

On the interval \([0, 1]\) the equilibrium temperature distribution is determined from
\[ u''(x) + 1 = 0, \quad u(0) = 0; \]
so
\[ u(x) = -\frac{x^2}{2} + C_1 x, \quad 0 \leq x \leq 1. \]

On the interval \([1, 2]\) one has \(u'' = 0\) and \(u(2) = 0\); so
\[ u(x) = C_2 (2 - x), \quad 1 \leq x \leq 2. \]
At the point \(x = 1\) the function must be continuous (perfect thermal contact), so
\[ C_1 - \frac{1}{2} = C_2. \]
The heat flow to the right at the point \(x = 1\) equals \(-u'(1^-)\) because \(K_0(1^-) = 1\), and it equals \(-2u'(1^+)\) because \(K_0(1^+) = 2\). Therefore, \(u'(1^-) = 2u'(1^+)\), or \(C_1 - 1 = -2C_2\). From the equations for \(C_1\) and \(C_2\) one gets \(C_1 = 2/3, C_2 = 1/6\).
Answer:
\[ u(x) = \begin{cases} 
-\frac{x^2}{2} + \frac{7}{6} x, & \text{when } 0 \leq x \leq 1 \\
\frac{1}{6} (2 - x), & \text{when } 1 \leq x \leq 2.
\end{cases} \]

Problem 1.4.11

(a) The total energy, as a function of time is given by
\[ E(t) = \int_0^L u(x, t)dt. \]
We differentiate \(E(t)\):
\[ E'(t) = \int_0^L u_t(x, t)dx = \int_0^L (u_t x(x, t) + x)dx = u_x(L, t) - u_x(0, t) + \frac{L^2}{2} = 7 - \beta + \frac{L^2}{2}. \]
The initial energy
\[ E(0) = \int_0^L f(x)dx. \]
Therefore,
\[ E(t) = \int_0^L f(x)dx + \left(7 - \beta + \frac{L^2}{2}\right)t. \]
(b) The equilibrium exists when \(\beta = 7 + L^2/2\). Then the equilibrium solution is
\[ u(x) = -\frac{x^3}{6} + C_1 x + C_2. \]
From the boundary conditions one gets \(C_1 = 7 + L^2/2\). To find the constant \(C_2\) one should use the condition that the energy of the equilibrium solution equals the initial energy:
\[ \int_0^L f(x)dx = \int_0^L u(x)dx = \frac{L^4}{24} + C_1 \frac{L^2}{2} + C_2 L = \frac{11}{24} L^4 + \frac{1}{2} L^2 + C_2 L. \]
Therefore
\[ C_2 = \frac{1}{L} \int_0^L f(x)dx - \frac{11}{24} L^3 - \frac{1}{4} L. \]
Answer:
\[ u(x) = -\frac{x^3}{6} + \left(7 + \frac{1}{2} L^2\right)x + \frac{1}{L} \int_0^L f(x)dx - \frac{11}{24} L^3 - \frac{1}{4} L. \]
Problem from the notes

Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of the heat equation $u_t = u_{xx}$ that satisfy the same initial data $u(x,0) = f(x)$ and the same boundary conditions $u_x(a,t) = \phi(t)$ and $u_x(b,t) = \psi(t)$. Let

$$u(x,t) = u_1(x,t) - u_2(x,t)$$

and

$$v(x,t) = u_x(x,t).$$

Then $v(x,t)$ satisfies the heat equation (that was the hint; $v(x,0) = 0$, $v(a,t) = 0$, and $v(b,t) = 0$. By the uniqueness theorem that was proved in the notes, $v(x,t) = 0$. That means that the function $u$ does not depend on $x$: its $x$-derivative vanishes; $u = u(t)$. From the heat equation, $u'(t) = 0$. Therefore, $u$ is a constant. On the other hand, $u(x,0) = 0$, so this constant equals 0. We conclude that $u = 0$. To summarize, any two solutions of the problem equal to each other. That means uniqueness.