Problem 4.4.3

(a) The term $-\beta u_t$ has a meaning of an external force; $u_t$ is the velocity. $\beta > 0$ means that the external force is directed opposite to the velocity, so it is a damping force indeed.

(b) We substitute $u(x, t) = g(t)\phi(x)$ into the equation

$$\rho_0 u_{tt} = T_0 u_{xx} - \beta u_t$$

to get

$$\rho_0 g''(t) \phi(x) = T_0 g(t) \phi''(x) - \beta g'(t) \phi(x).$$

We move the last term on the right to the left hand side of the equation, and we divide both sides by $T_0 g(t) \phi(x)$:

$$\frac{\rho_0}{T_0} g''(t) + \frac{\beta}{T_0} g'(t) = \frac{\phi''(x)}{\phi(x)}.$$

The expression on the left in the last equation does not depend on $x$, the expression on the right is independent of $t$, so both sides equal to a constant. We denote this constant by $-\lambda$. Then (1) splits into two equations:

(2) \hspace{1cm} \phi''(x) + \lambda \phi(x) = 0

and

(3) \hspace{1cm} \rho_0 g''(t) + \beta g'(t) + \lambda T_0 g(t) = 0.

The boundary conditions imply

$$\phi(0) = \phi(L) = 0,$$

so the solutions of (3) are

$$\lambda_n = \left( \frac{\pi}{L} \right)^2, \quad \phi_n(x) = \sin \left( \frac{\pi}{L} nx \right).$$

To solve (2), with $\lambda = \lambda_n$, we first solve the characteristic equation

$$\rho_0 \mu^2 + \beta \mu + \lambda_n T_0 = 0.$$

$$\mu_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4\rho_0 T_0 \lambda_n}}{2\rho_0} = -\frac{\beta}{2\rho_0} \pm i \omega_n$$

where

$$\omega_n = \sqrt{\frac{4\pi^2 \rho_0 T_0 \lambda_n}{L^2} - \beta^2}.$$
Under the condition $\beta^2 < 4\pi^2 \rho_0 T_0 / L^2$, all $\omega_n$'s are real, and the general solution of (3) is

$$ g_n(t) = e^{-\beta \rho_0 / \omega_n} (a_n \cos(\omega_n t) + b_n \sin(\omega_n t)). $$

Therefore, the solution of our problem can be looked for in the form

$$ u(x, t) = \sum_{n=1}^{\infty} e^{-\beta t / \rho_0} (a_n \cos(\omega_n t) + b_n \sin(\omega_n t)) \sin \left( \frac{\pi}{L} nx \right). $$

The initial conditions imply

$$ f(x) = \sum_{n=1}^{\infty} a_k \sin \left( \frac{\pi}{L} nx \right) $$

and

$$ g(x) = \sum_{n=1}^{\infty} \left( -\frac{\beta}{2 \rho_0} a_n + \omega_n b_n \right) \sin \left( \frac{\pi}{L} nx \right). $$

Therefore,

$$ a_n = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{\pi}{L} nx \right) \, dx $$

and

$$ b_n = \frac{\beta}{\rho_0 \omega_n L} \int_0^L f(x) \sin \left( \frac{\pi}{L} nx \right) \, dx + \frac{2}{\omega_n L} \int_0^L g(x) \sin \left( \frac{\pi}{L} nx \right) \, dx. $$

**Problem 4.4.5**

In the case when $4\pi^2 \rho_0 T_0 / L^2 < \beta < 16\pi^2 \rho_0 T_0 / L^2$ everything goes through as in the problem 4.4.3, with the only exception that $\omega_1$ is not real, but it is purely imaginary. Let

$$ \mu_{\pm} = -\beta \pm \sqrt{\beta^2 - 4\pi^2 \rho_0 T_0 / L^2}. $$

Then

$$ g_1(t) = a_+ e^{\mu_+ t} + a_- e^{\mu_- t} $$

and

$$ u(x, t) = (a_+ e^{\mu_+ t} + a_- e^{\mu_- t}) \sin \left( \frac{\pi}{L} x \right) $$

$$ + \sum_{n=2}^{\infty} e^{-\beta t / \rho_0} (a_n \cos(\omega_n t) + b_n \sin(\omega_n t)) \sin \left( \frac{\pi}{L} nx \right). $$

The initial conditions imply

$$ f(x) = (a_+ + a_-) \sin \left( \frac{\pi}{L} x \right) + \sum_{n=2}^{\infty} a_k \sin \left( \frac{\pi}{L} nx \right) $$
and
\[ g(x) = (\mu_+ a_+ + \mu_- a_-) \sin \left( \frac{\pi}{L} x \right) + \sum_{n=2}^{\infty} \left( -\frac{\beta}{2\rho_0} a_n + \omega_n b_n \right) \sin \left( \frac{\pi}{L} nx \right). \]

For \( a_n \) and \( b_n \), \( n \geq 2 \), one gets formulas (7) and (8). For \( a_{\pm} \), one gets a system of equations
\[ a_+ + a_- = \frac{2}{L} \int_0^L f(x) \sin \left( \frac{\pi}{L} x \right) dx \]
\[ \mu_+ + \mu_- = \frac{2}{L} \int_0^L g(x) \sin \left( \frac{\pi}{L} x \right) dx. \]
The solution of this system is
\[ a_+ = \frac{2}{L(\mu_+ - \mu_-)} \int_0^L (g(x) - \mu_- f(x)) \sin \left( \frac{\pi}{L} x \right) dx, \]
\[ a_- = \frac{2}{L(\mu_- - \mu_+)} \int_0^L (g(x) - \mu_+ f(x)) \sin \left( \frac{\pi}{L} x \right) dx. \]

**Problem 4.4.9**

One has
\[ E'(t) = \int_0^L (c^2 u_x u_{xt} + u_t u_{tt}) dx. \]
One uses the equation \( u_{tt} = c^2 u_{xx} \) to rewrite the integrand as \( c^2 (u_x u_{xt} + u_t u_{xx}) \).
Notice that the last expression is the partial derivative of \( c^2 u_x u_t \) with respect to \( x \). By the fundamental theorem of Calculus,
\[ \frac{dE(t)}{dt} = c^2 \int_0^L \frac{\partial (u_x u_t)}{\partial x} dx = c^2 (u_x(L, t) u_t(L, t) - u_x(0, t) u_t(0, t)). \]

**Problem 4.4.10**

(a) In this case, \( u_t(0, t) = u_t(L, t) = 0 \), so the right hand side in (9) vanishes, and the energy is constant.
(b) Now, \( u_x(0, t) = 0 \) and \( u_t(L, t) = 0 \); the right hand side of (9) vanishes, and the energy is constant.
(c) In this case, \( u_t(0, t) = 0 \), and
\[ E'(t) = -c^2 \gamma u(L, t) u_t(L, t). \]
Notice that
\[ u(L, t) u_t(L, t) = \frac{d}{dt} \left( \frac{u^2(L, t)}{2} \right). \]
Therefore,
\[ \frac{d}{dt} \left( E(t) + c^2 \gamma \frac{u^2(L, t)}{2} \right) = 0 \]
and
\[ E(t) + c^2 \gamma \frac{u^2(L, t)}{2} = \text{const.} \]
Because \( u^2(L, t) \geq 0 \), the energy \( E(t) \) remains bounded.
(d) In the case \( \gamma < 0 \), (10) does not imply that the energy is bounded; it can grow infinitely.