Symmetric Operator and their extensions

1. Let $A$ be a densely defined linear operator in a Hilbert space $H$. We say that a complex number $\lambda$ is a point of regular type for $A$ if

$$|| (\lambda I - A) x || \geq C ||x||, \quad x \in D(A)$$

(1)

where $C$ is a positive constant and $D(A)$ is the domain of $A$.

Notations. By $R_\lambda(A)$ I will denote the range of the operator $\lambda I - A$.

Proposition 1. Let $A$ be a closed operator, and let $\lambda$ be a point of regular type. Then $R_\lambda(A)$ is closed.

Proof. Let $y_n = \lambda x_n - Ax_n \to y$. Then $||x_n - x_m|| \leq ||y_n - y_m||/C$ (see (1)). Therefore, $\{x_n\}$ is a Cauchy sequence, and $x_n \to x$. On the other hand, $Ax_n \to \lambda x - y$. The operator $A$ is closed, so $x \in D(A)$ and $Ax = \lambda x - y$. We conclude that $y = (\lambda I - A)x \in R_\lambda(A)$.

QED

Exercise. Show that if $A$ is a closable operator and $\lambda$ is a point of regular type for $A$ then $R_\lambda(A) = R_\lambda(A)$.

Definition. Let $A$ be a closable operator in a separable Hilbert space $H$, and let $\lambda$ be a point of regular type for $A$. The codimension of the subspace $R_\lambda(A)$ in $H$ is called or the deficiency number of $\lambda$, and it will be denoted by $\text{def}(\lambda; A)$ or $m(\lambda; A)$.

Remark. For a closed operator $A$, $\lambda$ is a regular point iff it is a point of regular type and $m(\lambda; A) = 0$.

Theorem 2. Let $A$ be a closable operator in a separable Hilbert space $H$. The set of points of regular type of $A$ is open, and $m(\lambda; A)$ is constant on each connected components of this set.

Proof. 1. Let $\lambda$ be a point of regular type for $A$; so (1) holds. Take any point $\mu$ such that $|\mu - \lambda| < C/2$. Then

$$|| (\mu I - A) x || \geq || (\lambda I - A) x || - |\lambda - \mu| ||x|| \geq (C/2)||x||,$$

(2)

so $\mu$ is a point of regular type.

2. To prove that $m(\lambda; A)$ is constant on connected components of the set of points of regular type, one has to show that the function $m(\lambda; A)$ is continuous in $\lambda$. Let $\lambda$ be a point of regular type such that (1) holds, and let $|\mu - \lambda| < C/4$. Then (2) holds. We will show that $\dim R_\lambda(A) = \dim R_\mu(A)$.

Lemma 3. Let $L_1$ and $L_2$ be subspaces of a Hilbert space $H$, and let $P_j$ be the orthogonal projection onto $L_j$, $j = 1, 2$. If $||P_1 - P_2|| < 1$ then $\dim L_1 = \dim L_2$.

Proof. The restriction of $P_1$ to $L_2$ is a one-to-one mapping into. In fact, if $P_1 x = 0$ and $x \in L_2$, then $||(P_1 - P_2)x|| = ||x|| = ||x||$, and $x = 0$. Therefore, $\dim L_2 \leq \dim L_1$. Clearly, the opposite inequality also holds.

QED

By $\Theta(L_1, L_2)$ we denote the number $\sup_{x \in L_1 \setminus 0} ||x - P_2 x||/||x||$.

Exercise. Show that

$$||P_1 - P_2|| = \max\{\Theta(L_1, L_2), \Theta(L_2, L_1)\}.$$
To finish the prove of the theorem, it is sufficient to show that both \( \Theta(R_{\lambda}(A), R_{\mu}(A)) \) and \( \Theta(R_{\mu}(A), R_{\lambda}(A)) \) are smaller then 1. Indeed, the difference between the orthogonal projections onto \( R_{\lambda}(A) \perp \) and \( R_{\mu}(A) \perp \) is the same as the difference between orthogonal projections onto \( R_{\lambda}(A) \) and \( R_{\mu}(A) \). One has

\[
\Theta(R_{\lambda}(A), R_{\mu}(A)) = \sup_{x} \inf_{y} \frac{||x||}{||y||} \frac{||x - (\mu I - A)x||}{||y - (\lambda I - A)y||} \leq \sup_{x} \frac{||x||}{||x||} \leq 1/4.
\]

In the last inequality, to get an upper bound for the infimum over \( y \), I took \( y = x \). The number \( \Theta(R_{\mu}(A), R_{\lambda}(A)) \) can be estimated in a similar way.

QED

Exercise. Let \( A \) be a closed operator. Show that \( R_{\lambda}(A) \perp = \ker(\lambda I - A^*) \).

2. An operator \( A \) is called symmetric if \( (Ax, y) = (x, Ay) \) for every \( x, y \in D(A) \). Equivalently, \( A \subset A^* \). The last characterization implies that symmetric operators are always closable. Another useful characterization of the class of symmetric operators is given by the following proposition.

**Proposition 4.** An operator \( A \) is symmetric iff the quadratic form \( (Ax, x) \), \( x \in D(A) \), is real-valued.

**Proof.** The "only if" part is obvious. Let us prove the "if" part. Take \( x, y \in D(A) \). Then

\[
0 = \Im(A(x + iy), x + iy) = \Im((Ay, x) - (Ax, y)) = \Re((Ay, x) - (Ax, y)).
\]

Therefore,

\[
\Re(Ax, y) = \Re(Ay, x) = \Re(Ay, x) = \Re(x, Ay)
\]

for all \( x, y \in D(A) \). Replacing \( x \) by \( ix \), we conclude that \( \Im(Ax, y) = \Im(x, Ay) \), and \( (Ax, y) = (x, Ay) \).

QED

Let \( A \) be a symmetric operator. I will show now that all points outside the real axis are points of regular type for \( A \). Let \( \lambda = \sigma + i\tau \), \( \tau \neq 0 \). Then

\[
||x||^2 \geq \left||((\lambda I - A)x, x)||/||x|| \right|^2 \geq \left|\Im((\lambda I - A)x, x)||/||x|| \right| = |\tau||x||.
\]

I have used the fact that \( (Ax, x) \) is a real number. Theorem 2 implies that \( m(\lambda; A) \) is constant in the upper half plane: we denote this constant by \( m_+(A) \). It is also constant in the lower half plane: that constant is denoted by \( m_-(A) \). If there is at least one real point of regular type then the whole set of points of regular type is connected, and \( m_+(A) = m_-(A) \). Otherwise, these two numbers may be different.

From this point I will fix a complex number \( \lambda \) from the upper half plane. The next theorem tells us exactly what the domain of \( A^* \) is.

**Theorem 5.** Let \( A \) be a closed symmetric operator, and let \( \lambda \) be a non-real complex number. Then

\[
D(A^*) = D(A) + R_\lambda(A) \perp + R_\lambda(A) \perp.
\]

**Proof.** Both \( R_\lambda^\perp \) and \( R_\lambda^\perp \) are eigenspaces of \( A^* \) (the corresponding eigenvalues are \( \lambda \) and \( \lambda^* \)) so they lie inside of \( D(A^*) \). This proves the \( \supset \) inclusion.
Let } x \in D(A^*) \text{. The vector } \lambda x - A^* x \text{ can be decomposed as a sum of a vector from } R_\lambda(A) \text{ and a vector from } \ker(\lambda I - A^*); \text{ the last vector will be denoted by } (\lambda - \bar{\lambda}) z:

\lambda x - A^* x = \lambda y - A y + (\lambda - \bar{\lambda}) z.

Here } y \in D(A) \text{ and } A^* z = \bar{\lambda} z. \text{ The last equality implies}

A^*(x - y - z) = \lambda(x - y - z).

Therefore, } w = x - y - z \in R_\lambda(A)^+. \text{ What remains to be proved is that } x + y + z = 0, x \in D(A), A^* y = \bar{\lambda} y, A^* z = \lambda z \text{ implies } x = y = z = 0. \text{ Indeed, we apply the operator } A^* \text{ to the equation } x + y + z = 0 \text{ to get } Ax + \lambda y + \lambda z = 0. \text{ We subtract the last equality from } \lambda x + \lambda y + \lambda z = 0: (\lambda I - A)x + (\lambda - \bar{\lambda})y = 0. \text{ The terms in the last equality belong two two mutually orthogonal subspaces of } H. \text{ Therefore, both of them must vanish. In particular, } y = 0. \text{ Similarly, } z = 0.

\text{QED}

\textbf{Corollary.} A closed symmetric operator } A \text{ is self-adjoint iff } m_+(A) = m_-(A) = 0.

3. \text{ Let } B \text{ be a symmetric extension of } A. \text{ Then } A \subset B \subset B^* \subset A^*. \text{ In particular this means that}

\[ D(B) \subset D(A^*) = D(A) + R_\lambda(A)^+ + R_\lambda(A)^- \]

and, on this domain, } B \text{ acts like } A^*. \text{ Keeping in mind Proposition 4, to find possible domains of symmetric extensions of } A, \text{ we would like to know for which vectors } x \in D(A) \text{ the imaginary part of } (A^* x, x) \text{ vanishes. Let } x = x_0 + x_1 + x_2 \text{ where } x_0 \in D(A), A^* x_1 = \lambda x_1, \text{ and } A^* x_2 = \lambda x_2. \text{ A simple computation shows that}

\[ 3(A^* x, x) = 3\lambda (||x_2||^2 - ||x_1||^2). \]

Therefore, } x \text{ may belong to } D(B) \text{ only if}

\[ ||x_1|| = ||x_2||. \tag{3} \]

On the other hand, if (3) holds for every } x \in D(B) \text{ then the operator } B \text{ is symmetric by Proposition 4. Let } G = D(B) \cap (R_\lambda(A)^+ + R_\lambda(A)^-). \text{ Then } G \text{ is the graph of an isometry } V \text{ from a subspace } L \subset R_\lambda(A)^\perp \text{ into } R_\lambda(A)^\perp. \text{ We conclude that every symmetric extension of } A \text{ can be obtained is the following way. Let } V \text{ be an isometry from } L \subset R_\lambda(A)^\perp \text{ into } R_\lambda(A)^\perp. \text{ Let } \Gamma(V) \text{ be the graph of } V. \text{ Then one takes } D(B) = D(A) + \Gamma(V), \text{ and } B \text{ acts on its domain as } A^*.

\text{Let us compute } m_{\pm}(B). \text{ It is easy to see that } R_\lambda(B) = R_\lambda(A) + L \text{ and } R_{-\lambda}(B) = R_{-\lambda}(A) + VL. \text{ Therefore, } m_+(B) = \dim R_\lambda(A)^\perp / L \text{ and } m_-(B) = \dim R_{-\lambda}(A)^\perp / VL. \text{ In particular, } m_+(B) = 0, \text{ and } B \text{ is self-adjoint iff } L = R_\lambda(A)^\perp \text{ and } VL = R_{-\lambda}(A)^\perp. \text{ Such an isometry exists iff } m_+(A) = m_-(A). \text{ Let us formulate our conclusions as a theorem.}

\textbf{Theorem 6.} \text{ Let } A \text{ be a closed symmetric operator in a separable Hilbert space } H. \text{ It has a self-adjoint extension iff } m_+(A) = m_-(A). \text{ If this condition is met then every self-adjoint extension of } A \text{ can be obtained by the following construction. Fix a non-real complex number } \lambda, \text{ and let } V \text{ be an isometry from } R_\lambda(A)^\perp \text{ onto } R_{-\lambda}(A)^\perp. \text{ Then } D(A V) = D(A) + \Gamma(V) \text{ where } \Gamma(V) \text{ is the graph of } V.