Stability of Lagrangian Points: James Webb Space Telescope

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Main Science Themes:
- First Light and Reionization
- The Assembly of Galaxies
- Birth of Stars and Protoplanetary Systems
- Planetary Systems and the Origin of Life

Studies the Universe in infrared light
Introduction to Lagrange Points

Where in space would it be ideal to place a telescope guaranteeing stability with minimal energy?
Brief Overview of the Study of Planetary Motion

- Ptolemy studied the motion of planets in a geocentric ideology
- Copernicus came up with a heliocentric ideology
- Tycho Brahe observed the motion of planets
- Kepler then developed his laws of planetary motion
- Newton came up with his laws of gravity
Lagrange
1736-1813

Wrote "Le Problème Des Trois Corps" in 1772 in which he attempts to solve the three body problem, which is not solvable analytically.

"If the 'Académie' wishes to honor my work, it would be a great motivation to perfect my work; I don't lose hope to be able to form from my method a theory of the Moon as complete as we could ask from the still imperfect state of the analysis" – p230

\[ i = \frac{\text{parall. } \theta}{\text{parall. moy. } \phi} ; \]
The Problem

Lagrange points are where forces balance out between two masses.

We will be considering those concerning the Earth and the Sun.
Forces

Gravity - "Let A, B, C be the masses of three bodies that attract each other proportionally in mass and inversely in the square of the distances" - p231

Coriolis -

Centrifugal - a fictitious force that draws a rotating body away from the center of rotation
Why we want to test stability

- To minimize the work of the JWST to stay in orbit
- If the telescope would orbit the point if perturbed
- How long the telescope would remain in orbit at L2
- For stable observations
Calculating Lagrange Points

\[ F_g = \sum_{j=1, j \neq i}^{N} GM_j m_i \frac{\hat{r}_{ij}}{r_{ij}^2} \]
Stationary Frame

\[ \vec{F} = \frac{-GM_1 m (\vec{r} - \vec{r}_1)}{|\vec{r} - \vec{r}_1|^3} - \frac{-GM_2 m (\vec{r} - \vec{r}_2)}{|\vec{r} - \vec{r}_2|^3} \]

- We seek solutions to \( F = 0 \) that keep the relative locations of the three masses fixed in time
- These stationary solutions are our Lagrange points
Co-Rotating Frame

Origin at the center of mass and an angular velocity given by Kepler's law:

\[ \Omega = \sqrt{\frac{G(m_1 + m_2)}{R^3}} = \text{constant} \]

\[ \vec{F}_\Omega = \vec{F} - 2m \left( \Omega \times \frac{d\vec{r}}{dt} \right) - m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \]

Effective force in co-rotating frame
- Coriolis and Centrifugal force
Solving

\[
\begin{align*}
\vec{F}_\Omega &= \Omega^2 \left( x - \frac{\beta(x + \alpha R)R^3}{((x + \alpha R)^2 + y^2)^{3/2}} - \frac{\alpha(x - \beta R)R^3}{((x - \beta R)^2 + y^2)^{3/2}} \right) \hat{i} \\
&\quad + \Omega^2 \left( y - \frac{\beta y R^3}{((x + \alpha R)^2 + y^2)^{3/2}} - \frac{\alpha y R^3}{((x + \beta R)^2 + y^2)^{3/2}} \right) \hat{j}
\end{align*}
\]

\[
\begin{align*}
\vec{\Omega} &= \Omega \vec{k} \\
The vector \vec{r} &= x(t) \hat{i} + y(t) \hat{j} \\
The vector \vec{r}_1 &= -\alpha R \hat{i} \\
The vector \vec{r}_2 &= \beta R \hat{i}
\end{align*}
\]

\[
\begin{align*}
\alpha &= \frac{\mu_2}{\mu_1 + \mu_2} \\
\beta &= \frac{\mu_1}{\mu_1 + \mu_2}
\end{align*}
\]
Simplifications: L1, L2, L3

- Set m=1 without loss of generality
- Let x=R(u+B) so that u is the distance from the Earth in units of R
- Set y=0 and solve:

\[ u^2((1 - s_1) + 3u + 3u^2 + u^3) = \alpha(s_0 + 2s_0u + (1 + s_0 - s_1)u^2 + 2u^3 + u^4) \]

- \( \alpha \ll 1 \) which is a realistic approximation
The Three Cases to Consider

\[ u^2((1-s_1) + 3u + 3u^2 + u^3) = \alpha(s_0 + 2s_0u + (1 + s_0 - s_1)u^2 + 2u^3 + u^4) \]

3 cases are:

\[(s_0, s_1) = (-1, 1)\]
\[(s_0, s_1) = (1, 1)\]
\[(s_0, s_1) = (-1, -1)\]

\[s_0 = \text{sign}(u)\]
\[s_1 = \text{sign}(u+1)\]

In each case there is one real root thus giving us three Lagrange Points.
**L1, L2, and L3**

- To lowest order of alpha we find:

\[
\begin{align*}
L1 &: (R \left[1 - \left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right], 0) \\
L2 &: (R \left[1 + \left(\frac{\alpha}{3}\right)^{\frac{1}{3}}\right], 0) \\
L3 &: (-R \left[1 + \frac{5\alpha}{12}\right], 0)
\end{align*}
\]

- L1 and L2 = 1.5 million kilometers from earth
- L3: Slightly farther out than Earth's orbit, directly opposite the sun
L4, L5

- Resolve Force into Components:

\[ F_{\Omega}^{\perp} = \alpha \beta y \Omega^2 R^3 \left( \frac{1}{(x - R\beta)^2 + y^2)^{3/2}} - \frac{1}{(x + R\alpha)^2 + y^2)^{3/2}} \right) \]

\[ F_{\Omega}^{\parallel} = \Omega^2 \frac{x^2 + y^2}{R} \left( \frac{1}{R^3} - \frac{1}{(x - R\beta)^2 + y^2)^{3/2}} \right) \]

- Set components equal to zero and solve
L4, L5

\[ L4 : \left( \frac{R}{2} \left( \frac{M_1 - M_2}{M_1 + M_2} \right), \frac{\sqrt{3}}{2} R \right), \]

\[ L5 : \left( \frac{R}{2} \left( \frac{M_1 - M_2}{M_1 + M_2} \right), -\frac{\sqrt{3}}{2} R \right). \]

- Equidistant from sun and Earth, and symmetric about x-axis
How to Achieve Our Goals

2D N-body Simulation
- Earth, Sun, and JWST
- Assume circular orbits for planets
- Co-rotating frame
- Realistic initial conditions

\[ v_o = \frac{\pi r^2}{T} \]
Simulation

- Python and C/C++
- Fourth Order Hermite Integrator (Predictor-Corrector)
- Higher order terms for accuracy
- Jerk to update Acceleration

\[ j_i = G \sum_{j=1 \atop j \neq i}^N M_j \left[ \frac{v_{ji}}{r_{ji}^3} - 3 \frac{(r_{ji} \cdot v_{ji})r_{ji}}{r_{ji}^5} \right] \]
Testing Stability

- Validate physics by testing that the Earth orbits the Sun (assess energy budget)
- Get our co-rotating frame set up
- Test our analytic solutions
- Place the "space telescope" at those points in our code and let it run
- Slightly perturb initial conditions to see what happens to telescope
- Search for stability regions, classify fixed points and investigate what would happen after a longer period of time
Sources

http://www.nasa.gov/topics/universe/features/webb-l2.html

http://jwst.nasa.gov/launch.html


https://www.google.com/search?q=lagrange+point&espv=21O&es_sm=93&source=lnms&tbm=isch&sa=X&ei=9hsZU_HSMAWg7AYW21DIAQ&ved=0AcQ_AUoAg&biw=1366&bih=624&fsrc=&imgrc=XTNA_sHWTKGRuM%253A%3BfhX7r6q5mf6EuM%3Bhttp%253A%252F%252Fwww.labslices.net%252Fpictures%252Fblog%252F4cd8b19d3cc59128926961_blog.jpg%3Bhttp%253A%252F%252Fwww.labslices.net%252Fview_blog.php%252FblogID%2525D895%3B400%3B86

http://www-gap.dcs.st-and.ac.uk/history/PictDisplay/Lagrange.html