6.11a) Let $R$ be the relation on $\mathbb{N}$ given by $xRy$ iff $x$ divides $y$.

**Proposition 1** $R$ is reflexive and transitive, but not symmetric.

**Proof.** We check the properties individually:

- Reflexive: If $n \in \mathbb{N}$, then $n = (1)n$, so $n$ divides $n$. The relation is reflexive.
- Symmetric: This property is not satisfied. Notice that 2 divides 4 but 4 does not divide 2, so $2R4$ but $4 \not\in R2$.
- Transitive: Let $x, y, z \in \mathbb{N}$ such that $xRy$ and $yRz$, so there are integers $m, n$ such that $y = mx$ and $z = ny$. Thus there is $z = (mn)x$, so $x$ divides $z$ and $xRz$. Thus the relation is transitive.

6.11b) Let $X$ be a set and let $R$ be the relation “$\subseteq$” defined on subsets of $X$.

**Proposition 2** $R$ is reflexive and transitive, but not symmetric unless $X = \emptyset$.

**Proof.** We check the properties individually:

- Reflexive: If $A \subseteq X$, then $A \subseteq A$, so $ARx$. Thus $R$ is reflexive.
- Symmetric: This property is not satisfied unless $X$ is the empty set, since we see that $\emptyset \subseteq X$, but $X \not\subseteq \emptyset$, so $\emptyset RX$ and $X \not\in R\emptyset$. If $X$ is the empty set, then it is clearly true.
- Transitive: Let $A, B, C$ be subsets of $X$. Suppose $ARB$ and $BRC$. Then $A \subseteq B$ and $B \subseteq C$. It follows that $A \subseteq C$ since if $x \in A$, then $x \in B$ then $x \in C$. Thus $ARC$. Thus the relation is transitive.

6.20) Let $R$ be a relation on $\mathbb{Z}$ defined by $xRy$ iff $x - y = 3k$ for some integer $k$.

**Proposition 3** $R$ is an equivalence relation.

**Proof.** We check the properties of an equivalence relation individually:

- Reflexive: Let $x \in \mathbb{Z}$. Then $x - x = 0 = 3(0)$, so $xRx$.
- Symmetric: Let $x, y \in \mathbb{Z}$ such that $xRy$. Thus there exists $k \in \mathbb{Z}$ such that $x - y = 3k$. It follows that $y - x = 3(-k)$, so $yRx$.
- Transitive: Let $x, y, z \in \mathbb{Z}$ such that $xRy$ and $yRz$. Thus there exist $k$ and $k'$ in $\mathbb{Z}$ such that $x - y = 3k$ and $y - z = 3k'$. We then see that

$$x - z = (x - y) + (y - z) = 3k + 3k' = 3(k + k').$$

Thus $xRz$. 


The equivalence class \([5] = E_5\) consists of all integers equivalent to 5, i.e., all \(x \in \mathbb{Z}\) such that \(5 - x = 3k\) for some integer \(k\). Thus
\[
[5] = \{5 - 3k : k \in \mathbb{Z}\} = \{2 + 3k : k \in \mathbb{Z}\}.
\]

We note that there are three equivalence classes: \([0]\), \([1]\), \([2]\), since we can always see that, for \(x \in \mathbb{Z}\),
\[
[x] = \left[x - 3 \left(\frac{x}{3}\right)\right].
\]

\([x]\), “the floor of \(x\),” denotes the largest integer less than or equal to \(x\).) Notice that
\[
\frac{x}{3} - 1 < \left[\frac{x}{3}\right] \leq \frac{x}{3},
\]
so
\[
-x \leq -3 \left[\frac{x}{3}\right] < 3 - x,
\]
and
\[
0 \leq x - 3 \left(\frac{x}{3}\right) < 3.
\]

Thus \([x]\) always equals \([0]\) or \([1]\) or \([2]\).

6.25) Define a relation \(R\) on \(\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})\) by \((a, b) R (x, y)\) iff \(ay = bx\).

**Proposition 4** \(R\) is an equivalence relation.

**Proof.** We prove each property individually:

- **Reflexive:** For any \((a, b) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})\), we have that \(ab = ba\), so \((a, b) R (a, b)\).

- **Symmetric:** Suppose \((a, b) R (x, y)\). Then \(ay = bx\), so \(xb = ya\) and \((x, y) R (a, b)\).

- **Transitive:** Suppose \((a, b) R (c, d)\) and \((c, d) R (e, f)\). Then \(ad = bc\) and \(cf = de\). Thus, since \(d \neq 0\),
\[
af = \frac{adf}{d} = \frac{bce}{d} = \frac{be}{d}.
\]

Thus, \((a, b) R (e, f)\).

The equivalence classes are in correspondence with rational numbers in the following sense. The ordered pair \((a, b) R (x, y)\) if and only if \(\frac{a}{b} = \frac{x}{y}\), i.e., they represent the same rational number. Thus each equivalence class consists of all possible representations of rational numbers as a quotient of integers, and the set of all equivalence classes corresponds to the set of all rational numbers.