Math 323: Homework 9 Solutions

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7.12a) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \). We defined \( f + g \) by \((f + g)(x) = f(x) + g(x)\). It is not true that if \( f \) and \( g \) are bijective, then the sum \( f + g \) is bijective. Consider \( f(x) = x \) and \( g(x) = -x \). Both are clearly bijective (they are their own inverses). But the sum \( f + g \) is equal to the constant function zero, which is clearly not bijective.

7.12b) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \). We defined \( fg \) by \((fg)(x) = f(x)g(x)\). It is not true that if \( f \) and \( g \) are bijective, then their product \( fg \) is bijective. Consider \( f(x) = g(x) = x \). Then \( f(x)g(x) = x^2 \), which is neither injective nor surjective.

7.20) Suppose \( f : A \rightarrow B \) and suppose \( C \subseteq A \) and \( D \subseteq B \).

a) The statement \( f(C) \subseteq D \) iff \( C \subseteq f^{-1}(D) \) is true. Suppose \( f(C) \subseteq D \) and consider \( x \in C \). Then \( f(x) \in D \), so \( x \in f^{-1}(D) \). Conversely, suppose \( C \subseteq f^{-1}(D) \), and let \( y \in f(C) \). Thus there exists \( x \in C \) such that \( y = f(x) \). Since \( x \in f^{-1}(D) \), we know that \( f(x) \in D \), i.e., \( y \in D \).

b) If \( f \) is injective and \( D \subseteq \text{rng}(f) \) (in particular, if \( f \) is bijective), then \( f(C) = D \) iff \( C = f^{-1}(D) \).

**Proof.** By part (a), \( f(C) \subseteq D \) iff \( C \subseteq f^{-1}(D) \), thus we need that if \( f \) is injective, then \( D \subseteq f(C) \) iff \( f^{-1}(D) \subseteq C \). First suppose that \( D \subseteq f(C) \). Let \( x \in f^{-1}(D) \). Then \( f(x) \in D \), and hence \( f(x) \in f(C) \), i.e., there exists \( x' \in C \) such that \( f(x') = f(x) \). Since \( f \) is injective, \( x = x' \), so \( x \in C \).

Conversely, suppose \( f^{-1}(D) \subseteq C \). Let \( y \in D \). Since \( D \subseteq \text{rng}(f) \), there exists \( x \in A \) such that \( f(x) = y \). Since \( x \in f^{-1}(D) \), \( x \in C \). Thus \( y = f(x) \in f(C) \).

7.30) Suppose \( g : A \rightarrow C \) and \( h : B \rightarrow C \). If \( h \) is bijective, then there exists a function \( f : A \rightarrow B \) such that \( g = h \circ f \).

**Proof.** Since \( h \) is bijective, there is a function \( h^{-1} : C \rightarrow B \). If we define \( f \) to be \( h^{-1} \circ g \), then

\[
h \circ f = h \circ h^{-1} \circ g = \text{id}_C \circ g = g.
\]
Extra problem:
Consider the following relation on $[-\pi, \pi]$:

$x \sim y$ iff $x = y$ or $x, y \in \{-\pi, \pi\}$.

a) We first show that this is an equivalence relation. Notice that it is reflexive, since $x = x$. It is symmetric, since if $x \sim y$ then $x = y$ or $x, y \in \{-\pi, \pi\}$, which is the same as if we switch $x$ and $y$. To prove transitivity, we first assume $x \sim y$ and $y \sim z$. We consider two cases. First suppose that $x \notin \{-\pi, \pi\}$. Then $y = x$, and so $y \notin \{-\pi, \pi\}$. Thus $z = y = x$ and $x \sim z$. Now suppose $x \in \{-\pi, \pi\}$, then we must have $y \in \{-\pi, \pi\}$ and hence $z \in \{-\pi, \pi\}$, thus $x \sim z$.

b) We describe the equivalence classes. There is one equivalence class for every number between $-\pi$ and $\pi$ and a single equivalence class containing $\pi$ and $-\pi$. One can think of this as taking the interval $[-\pi, \pi]$ and gluing one end to the other.

c) Show that

$$f([x]) = (\cos x, \sin x)$$

is a well-defined function $A \to \mathbb{R}^2$, where $A$ is the set of equivalence classes of $\sim$.

We need to show that if $x \sim y$, then $(\cos x, \sin x) = (\cos y, \sin y)$. Certainly if $x = y$ this is true. Now suppose $x = \pi$ and $y = -\pi$. We see that $(\cos \pi, \sin \pi) = (-1, 0) = (\cos (-\pi), \sin (-\pi))$. Since $\sim$ is symmetric, this is sufficient.

d) Show that if we set the codomain to be

$$B = \{(\cos t, \sin t) \in \mathbb{R}^2 : t \in \mathbb{R}\},$$

then $f : A \to B$ is a bijection. Hint: you can use inverse trig functions, but be careful of where they exist and what their domains and ranges are!

First we show that $f$ is injective. Suppose $f([x]) = f([y])$, so $\cos x = \cos y$ and $\sin x = \sin y$. For $x \in [-\pi, \pi]$, $\cos x = \cos y$ only if $x = \pm y$. If $x = y$, we are done. Suppose $x = -y$. Then we see that $\sin x = \sin y = \sin(-x) = -\sin x$. So we must have $\sin x = 0$, so $x = \pm \pi$. Thus $[x] = [y]$. We now show that $f$ is surjective. Since $\cos t$ and $\sin t$ are periodic with period $2\pi$, for any $t \in \mathbb{R}$, if we let

$$x = t - 2\pi \left\lfloor \frac{t}{2\pi} + \frac{1}{2} \right\rfloor,$$

we have $\sin x = \sin t$, $\cos x = \cos t$, and

$$-\pi = t - 2\pi \left( \frac{t}{2\pi} + \frac{1}{2} \right) \leq t - 2\pi \left\lfloor \frac{t}{2\pi} \right\rfloor < t - 2\pi \left( \frac{t}{2\pi} - \frac{1}{2} \right) = \pi,$$

so $f([x]) = (\cos t, \sin t)$ and $f$ is surjective.

8.3c)

Proposition 1 The sets $S = [0, 1)$ and $T = (0, 1)$ are equinumerous.

Proof. We can always insert one element into an infinite set by taking out a countable set and shifting it. We can define the following function $f : T \to S$ by

$$f(x) = \begin{cases} 
    x & \text{if } x \neq \frac{1}{n+1} \text{ for all } n \in \mathbb{N} \\
    \frac{1}{n} & \text{if } x = \frac{1}{n+1} \text{ for some } n \in \mathbb{N}\setminus\{1\} \\
    0 & \text{if } x = \frac{1}{2}
\end{cases}$$

One can show it is bijective directly, or see that it has an inverse function $g : S \to T$ defined by

$$g(y) = \begin{cases} 
    y & \text{if } y \neq \frac{1}{n+1} \text{ for all } n \in \mathbb{N} \text{ and } y \neq 0 \\
    \frac{1}{n+2} & \text{if } y = \frac{1}{n+1} \text{ for some } n \in \mathbb{N} \\
    0 & \text{if } y = 0
\end{cases}$$

8.3e)
Proposition 2 The sets $S = (0, 1)$ and $T = \mathbb{R}$ are equinumerous.

Proof. There are several ways to get a bijection. The key idea is that we need a function that takes a finite interval to an infinite interval. These are functions like $\tan x$, $\frac{1}{x}$, and rational numbers. Also, one can compose several bijections together to get a bijection with the appropriate domain. So, for instance, $f(x) = \tan x$ gives a bijection between $(-\frac{\pi}{2}, \frac{\pi}{2})$ and $\mathbb{R}$, and so one can then scale and shift by precomposing with the function $g(x) = \pi x - \frac{\pi}{2}$ to get the function $\tan (\pi x - \frac{\pi}{2})$. Here are some bijections $S \rightarrow T$:

$$f_1(x) = \tan \left(\pi x - \frac{\pi}{2}\right)$$
$$f_2(x) = \frac{x - \frac{1}{2}}{x(1-x)}$$
$$f_3(x) = \begin{cases} \frac{1}{x - \frac{1}{2}} & \text{if } x \in (0, \frac{1}{2}) \\ \frac{1}{x - \frac{1}{2}} & \text{if } x \in (\frac{3}{4}, 1) \\ 16x - 8 & \text{if } x \in \left[\frac{1}{4}, \frac{3}{4}\right] \end{cases}$$

One can prove these are bijections by considering regions where they are increasing and decreasing, or sometimes by finding inverse functions (for instance, the inverse to $f_2$ is $g_3(y) = \begin{cases} \frac{1}{y} + \frac{1}{2} & \text{if } y < -4 \\ \frac{y}{y} + \frac{1}{2} & \text{if } y > 4 \\ \frac{1}{6}(y + 8) & \text{if } x \in [-4, 4] \end{cases}$.)

More Solutions:

7.13b) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$. Defined $fg$ by $(fg)(x) = f(x)g(x)$. It is not true that if $f$ and $g$ are bijective, then $fg$ is bijective. Consider $f(x) = g(x) = x$. Then both functions are bijective, but $(fg)(x) = x^2$, which is not bijective on $\mathbb{R}$.

8.4)

Proposition 3 Suppose $m < n$. Then $(0, 1)$ and $(m, n)$ are equinumerous.

Proof. The function $f : (0, 1) \rightarrow (m, n)$ defined by $f(x) = m + nx$ is a bijection if $n \neq 0$. If $n = 0$, then we can take $f(x) = n - mx$, and this is a bijection. (A linear function is injective if the slope is not zero. By looking at the image, we see that the function is a bijection.)

Proposition 4 Any two open intervals $(a_1, b_1)$ and $(a_2, b_2)$ are equinumerous.

Proposition 5 Since we have bijections $f_1 : (0, 1) \rightarrow (a_1, b_1)$ and $f_2 : (0, 1) \rightarrow (a_2, b_2)$, the map $f_2 \circ f_1^{-1} : (a_1, b_1) \rightarrow (a_2, b_2)$ is a bijection (the composition of bijections is a bijection).

8.5)

Proposition 6 If $S \setminus T \sim T \setminus S$ then $S \sim T$.

Proof. Suppose there is a bijection $f : S \setminus T \rightarrow T \setminus S$. Since we have that $S = (S \setminus T) \cup (S \cap T)$ (this follows quickly since if $x \in S$, then $x \in S$ and $x \notin T$ or else $x \in S$ and $x \in T$). Thus we can construct the function $g : S \rightarrow T$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in S \setminus T \\ x & \text{if } x \in S \cap T \end{cases}$$

Note that this is well-defined since the sets $(S \setminus T)$ and $(S \cap T)$ are disjoint. The function $g$ is surjective since if $x \in T$ then $x \in S \cap T$ or $x \in T \setminus S$. In the first case $g(x) = x$ and in second case, $g(f^{-1}(x)) = x$. Thus, $g$ is surjective. To prove injectivity, suppose $g(x) = g(x')$. Then if $g(x) \in T \setminus S$, then $x = f^{-1}(g(x)) = f^{-1}(g(x')) = x'$. If $g(x) \in S \cap T$ then $x = g(x) = g(x') = x'$. Thus $g$ is a bijection.
8.10) If $S$ is denumerable, then $S$ is equinumerous with a proper subset of itself.

**Proof.** Since $S$ is denumerable, there exists a bijection $f : \mathbb{N} \to S$. Consider the set $T = S \setminus \{f(1)\}$. Then consider the function $g : \mathbb{N} \to T$ given by $g(n) = f(n + 1)$. It is injective, since if $g(m) = g(n)$, then $f(m + 1) = f(n + 1)$, and since $f$ is injective, $m + 1 = n + 1$. Thus $m = n$. Furthermore, it is surjective, since for any $x \in T$, $x \in S$, so $x = f(n)$ for some $n > 1$. Thus $x = f(m + 1)$ for some $m \in \mathbb{N}$ ($m = n - 1$), i.e., $x = g(m)$. Thus $T$ is denumerable, and equinumerous to $S$. \qed