1. (a) Show that every field $F$ has the cancellation property, i.e., for every $a, x, y \in F$, $a \neq 0$

   $ax = ay \Rightarrow x = y.$

2. On $\mathbb{Z}_5$ we define $x\hat{+}y = x + y$ and $x \cdot y = x \cdot y$.

   (a) Show that $\hat{+}$ and $\cdot$ are well defined operations on $\mathbb{Z}_5$.

   (b) Show that $(\mathbb{Z}_5, \hat{+}, \cdot)$ is a field.
(c) Show that for every $n \in \mathbb{Z}_5^\times$ (i.e., $n \neq 0$), $n^4 = 1$.

3. Show that $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a field.
4. (a) On $\mathbb{Q}[\sqrt{2}]$ we define the relation: $a + b\sqrt{2} \prec a' + b'\sqrt{2}$ if $a < a'$ and $b < b'$. Show that $\prec$ is anti-reflexive, anti-symmetric, and transitive.

(b) Show that the field $\mathbb{Q}[\sqrt{2}]$ cannot be ordered (i.e., show that $\prec$ does not constitute a total order on $\mathbb{Q}[\sqrt{2}]$).
5. (a) Let $\mathbb{F}$ be a field. Show that for every $x \in \mathbb{F}$

$$(1 - x) (1 + x + x^2 + \ldots + x^{n-1}) = 1 - x^n.$$  

(b) Use the result in part (a) to show that for every $x, y \in \mathbb{F}$,

$$x^n - y^n = (x - y) \left( x^{n-1} + x^{n-2} y + \ldots + xy^{n-2} + y^{n-1} \right).$$
6. Consider the sequence \( x_n = \frac{1}{2} x_{n-1} + 3, \ x_1 = 1. \)

(a) Show that \( \forall n \in \mathbb{N}, \ x_n \leq 6. \)

(b) Show that \( (x_n)_{n \in \mathbb{N}} \) is a nondecreasing sequence.

(c) Use parts (a) and (b) to conclude that \( (x_n)_{n \in \mathbb{N}} \) must converge to a limit. What is this limit?
7. (a) We propose to find an explicit formula for $x_n$ in problem 6. Show that $x_{n+1} - x_n = \frac{1}{2} (x_n - x_{n-1})$.

(b) Use the result in part (a) to show that $x_{n+1} - x_n = \frac{5}{2^n}$.

(c) Find an explicit formula for $x_n$.

(d) Show that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, such that:

\[ n, m \geq N \Rightarrow |x_m - x_n| < \epsilon. \]
8. (a) Find the multiplication inverse of 17 in $\mathbb{Z}_{29}$.

(b) Solve in $\mathbb{Z}_{13}$, the equation $2x^3 + 4x^2 + x + 2 = 0$

(c) Find all the integers $x, y$ for which $23x + 15y = 1$. 