19/22/09  (TUESDAY AFTERNOON)

IDEA: Given recent Matlab lab numerically estimating integrals via Riemann sums as well as geometric interpretation of the definite integral as an area under a curve, now seems like a natural time to motivate using integrals to estimate areas + volumes (H.H. Ch. 8.1)

(starter point)

1. Area underneath a curve

\[ \text{area} = \int_a^b f(x) \, dx \]

\[ \text{sum of lots of little guys} \]

[Diagram of area under curve]

2. Area between two curves

\[ \text{area} = \int_a^b f(x) \, dx - \int_a^b g(x) \, dx \]

\[ = \int_a^b [f(x) - g(x)] \, dx \]

[Diagram of area between two curves]
3. Extending out into another dimension (i.e., area → volume)

\[ \text{volume enclosed} = \text{length} \times \text{height} \times \text{width} = \int_a^b f(x) \, dx \]

4. Arbitrary Geometric Shapes (2-D)

5. Suppose you aren't given a nice curve \( f(x) \) per se as in the previous example, but you need to set up the geometry yourself (e.g., determine a suitable coordinate system).

6. Find the area of a triangle using Riemann sums.
   [Note: we already know the answer! \( \text{area} = \frac{1}{2} \text{(base)} \times \text{(height)} \)]

7. Create a spatial axis (only one dimension needed), here given the symmetry, and determine an orientation and scale.

8. Consider an infinitesimal strip of width \( \Delta x \) (where \( \Delta x \) is small such that the error due to using a rectangle across that particular slice is small).

\[ \Delta A = \text{area of strip} = \text{width} \times \Delta x \]  

\[ \text{where width} = w(x) \]

9. Consider the geometry such that you can specify \( w(x) \) exactly.
- We know that \( w(0) = b \) and \( w(h) = 0 \).

- Given that the sides of the triangle are straight, \( w(x) \) will vary linearly along \( x \) \( \rightarrow w(x) = kx + c \).

- Now plug in our conditions specified above to find \( k \) and \( c \):

\[
\begin{align*}
  w(0) = b &= k(0) + c \quad \rightarrow \quad c = b, \\
  w(h) = 0 &= kh + b \quad \rightarrow \quad k = -\frac{b}{h}.
\end{align*}
\]

\( \Rightarrow \) Now we have an expression for the width of a strip at any given height \( x \) and thus the area of the strip:

\[
\Delta A = w(x) \cdot \Delta x = \left( -\frac{b}{h} x + b \right) \Delta x
\]

- Now if we add up all the strips, we should have a good approximation of the area; in the limit where the height of the strip gets infinitesimally small (i.e., \( \Delta x \to 0 \)) or put in other words, we have an infinite # of strips then this will become exact.

\[
A \approx \sum_{x=0}^{b} \Delta A = \sum_{x=0}^{b} \left( -\frac{b}{h} x + b \right) \Delta x
\]

\[
= \lim_{\Delta x \to 0} \sum_{x=0}^{b} \left( -\frac{b}{h} x + b \right) \Delta x = \int_{0}^{b} \left( -\frac{b}{h} x + b \right) dx
\]

\[
= -\frac{b}{h} \left[ \frac{1}{2} x^2 \right]_{0}^{b} + b(x) \mid_{0}^{b} = -\frac{1}{2} bh + bh = \frac{1}{2} bh \quad (1)
\]

\( \Rightarrow \) Though we took a more rigorous approach, we ended up w/ what we know to be the correct answer!
Problem: Find the volume of a circle with radius \( r \).

\[ \Delta A = (r^2 - x^2) \Delta x \]

Consider the geometry, i.e., a right triangle, such that we must always have:

\[ x^2 + h^2 = r^2 \]

Thus, \( h = h(x) = \sqrt{r^2 - x^2} \).

Thus, the area of our strip becomes:

\[ \Delta A = 2(r^2 - x^2)^{\frac{1}{2}} \Delta x \]

Similar to before, we just add up all possible strips (by taking a limit).

\[ A = \lim_{\Delta x \rightarrow 0} \sum \Delta A = \int_{-r}^{r} 2(r^2 - x^2)^{\frac{1}{2}} \, dx \]

\[ = 2 \int_{-r}^{r} \sqrt{r^2 - x^2} \, dx \]

\[ = 2 \int_{-r}^{r} \sqrt{1 - \left( \frac{x}{r} \right)^2} \, dx \]

Let \( \frac{x}{r} = \sin \theta \)

\[ = 2r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^2 \theta}{r^2} \, \theta \]

\[ = \pi r^3 \]

\( \text{(since } \sin^2 \theta + \cos^2 \theta = 1, \ 1 - \sin^2 \theta = \cos^2 \theta \) \]

(\( \text{limits change too!} \)}

\( x = r \sin \theta \rightarrow \theta = \frac{\pi}{2} \)
so we need to solve an integral of the form:
\[ \int \cos^2 \theta \, d\theta \]
we could do this in a # of ways:
1. numerically (via your Matlab Code.)
2. table of integrals
3. integration by parts
   ...we will flesh these out over the next few classes

\[ \int \cos^2 \theta \, d\theta = \frac{1}{2} \cos \theta \sin \theta + \frac{1}{2} \theta + C \]

→ for the sake of time, we will simply take as given:

\[ \int \cos^2 \theta \, d\theta \]

→ back to our problem:

\[ A = 2r^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 \theta \, d\theta = 2r^2 \left[ \frac{1}{2} \left( \cos \theta \sin \theta + \theta \right) \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \]

\[ = r^2 \left[ \cos \left( \frac{\pi}{2} \right) \sin \left( \frac{\pi}{2} \right) + \frac{\pi}{2} - \left( \cos \left( -\frac{\pi}{2} \right) \sin \left( -\frac{\pi}{2} \right) + \left( -\frac{\pi}{2} \right) \right) \right] \]

\[ = r^2 \left[ \frac{\pi}{2} - \frac{-\pi}{2} \right] = \pi r^2 \]

→ again, once the smoke cleared, we ended up with an answer we knew we should end up with!

Volumes

→ now apply these same ideas to volumes, taking 'slices' rather than 'strips'

Cylinder

Consider horizontal slices...
(easy)

\[ h \]

\[ \text{or vertical slices \ (harder, but similar to approach we did above)} \]

Cone

\[ \text{Strategize to choose the easiest 'slicing paradigm'} \]