Existence + Uniqueness Theorem
(first order ODEs) \[ \frac{dy}{dx} = g(x, y), \quad y(x_0) = y_0. \]

Implementation (see L.L. pg. 57) solution exists?
if so, is it 'unique'?

1. Confirm that \( g(x, y) \) is continuous in the vicinity of \((x_0, y_0)\).

2. Compute \( \frac{\partial g}{\partial y} \) and is cont. "...

3. If \#1 and \#2, then a unique solution exists. If not, we are not sure (i.e., there may or may not be a solution, that is or is not unique).

Partial Derivatives \( g(x, y) \rightarrow \) function of more than one variable!

Rule: to differentiate, keep one variable 'const.'

Example \( g(x, y) = (xy)^{\frac{1}{2}} \rightarrow \frac{\partial g}{\partial y} = \frac{1}{2} \sqrt{\frac{x}{y}} \)

\[ g = x^{\frac{1}{2}} y^{\frac{1}{2}} \rightarrow \frac{\partial}{\partial y} g = x^{\frac{1}{2}} \frac{\partial}{\partial y} y^{\frac{1}{2}} = x^{\frac{1}{2}} \cdot \frac{1}{2} y^{-\frac{1}{2}} = \frac{1}{2} \sqrt{\frac{x}{y}}. \]

\[ \frac{\partial g}{\partial x} = \frac{1}{2} \sqrt{\frac{y}{x}}. \]
\[
\frac{2}{\sqrt{y}} \frac{\partial}{\partial y} \left( \frac{3y}{2x} \right) = \frac{2}{\sqrt{y}} \frac{1}{2} \frac{1}{x} y^{-\frac{1}{2}} = \frac{1}{2} \sqrt{y} \frac{dy}{dx} = \frac{1}{4} y^{-\frac{1}{2}} \left( \frac{1}{x} \right) = \frac{1}{4} \sqrt{\frac{1}{xy}}
\]

\[
\frac{\partial}{\partial y} \left( \frac{\partial}{\partial x} \right) = \frac{1}{4} \sqrt{\frac{1}{xy}}
\]

\[\text{Ex} \quad \frac{\partial}{\partial a} (a, b) = ab^2 + 3a^2 e^b \quad \implies \quad \text{find } \frac{\partial}{\partial a} \left( \frac{\partial}{\partial a} \right)
\]

\[
\frac{\partial}{\partial a} \frac{\partial}{\partial a} = \frac{2}{\partial a} \left( ab^2 + 3a^2 e^b \right) = 2b + 6ae^b
\]

\[
\frac{\partial}{\partial b} \frac{\partial}{\partial a} = \frac{2}{\partial b} \left( b^2 + 6ae^b \right) = 2b + 6ae^b
\]

Exist. + Uniq. Examples

\[\frac{dy}{dx} = y^3 \quad y(0) = 0\]

\[y^3 \text{ is cont. in vicinity of } y(0) = 0\]

(Actually cont. everywhere!)

\[\frac{\partial y}{\partial x} = 3y^2 \quad \text{cont. everywhere}\]

\[\implies \text{there is a unique solution to } \frac{dy}{dx} = y^3 \text{ that passes through the point } y(0) = 0\]
Given \( \frac{dy}{dx} = (xy)^{\frac{1}{2}} \), \( y(1) = 1 \)

- \( g(x, y) \) is cont. when \( \begin{cases} 
  x = y = 0 \\
  x, y \geq 0 \\
  x, y \leq 0
\end{cases} \)

- \( \frac{\partial g}{\partial y} = \frac{1}{8} \sqrt{\frac{x}{y}} \quad \rightarrow \text{cont. when} \quad \begin{cases} 
  x > 0 \\
  x, y < 0
\end{cases} \)

- \( \text{Cont. when} \quad \begin{cases} 
  x \geq 0 \text{ AND } y > 0 \\
  x \leq 0 \text{ AND } y < 0
\end{cases} \)

\( \rightarrow \) a unique solution exists for \( y(1) = 1 \)

\( y(-2) = +1 \quad \rightarrow \text{not sure if a (unique) solution exists} \)

**Exercise:** True or false: There are at least three different solutions to \( \frac{dy}{dx} = y^{\frac{1}{3}} \) passing through \((0,0)\)