PROBLEM SOLVING RND II. (see online notes)

11/6/09  250A lecture (L1, L2)

**Goal**: solve \( y' = \frac{x^2 + xy + y^2}{xy} \) \( \rightarrow \) highly nonlinear, not separable

\[ \Rightarrow \text{how to approach? can we simplify?} \]

**Aside**: consider an eqn. of the form \( y' = g \left( \frac{y}{x} \right) \)

\[ \Rightarrow \text{so } g \text{ does depend upon } x \text{ and } y, \text{ but in a way such that} \]
\[ \text{it is the ratio of } y \text{ to } x \text{ that determines its form} \]
\[ \text{(we call this an ODE w/ HOMOGENEOUS COEFFICIENTS)} \]

let \( z = \frac{y}{x} \), then \( y = zx \)

then
\[ y' = \frac{dy}{dx} = \frac{d}{dx} (xz) = z + x \frac{dz}{dx} = z + xz' \]

back to the ODE: \( y' = g \left( \frac{y}{x} \right) = g(z) = z + xz' \)

\[ \Rightarrow \frac{dz}{z} = \frac{g(z) - z}{x} \]

which is actually separable!!

So now we need to solve (upon separating variables):

\[ \int \frac{dz}{g(z) - z} = \int \frac{1}{x} \, dx \]

\[ \Rightarrow \text{hopefully from here, we can solve for } z(x) \]
\[ \text{and thereby } y(x) = x \cdot z(x). \]
→ back to our problem: \[ y' = \frac{x^2 + xy + y^2}{xy} \]

**QUESTION:** Is this ODE homogeneous [i.e., can we write it as \( y' = g(x) \)]?

→ to determine this, set \( y = ax \) (where \( a \) is a const.):

\[ \frac{x^2 + xy + y^2}{xy} = \frac{x^2 + ax^2 + a^2x^2}{x^2a} = \frac{1 + a + a^2}{a} \]

→ if \( g(x, ax) \) does not depend upon \( x \), then \( g \) is homogeneous of degree 0 and we can apply the method of homogeneous coefficients.

letting \( z = \frac{y}{x} \), → \( y' = \frac{x^2 + xy + y^2}{xy} = \frac{z}{y} + 1 + \frac{z}{x} \)

\[ \frac{z'}{x} + z = \frac{1}{z} + 1 + \frac{z}{x} \]

→ \[ z' = \frac{1}{x} \left( \frac{1}{z} + 1 \right) \]

→ which we know how to solve!!

\[ \int \frac{1}{x} \, dx = \int \frac{1}{z} \, dz = \int \frac{z}{1+z} \, dz = \int \frac{\frac{z}{1+z}}{1+z} \, dz \]

\[ = \int \left( 1 - \frac{1}{z+1} \right) \, dz = \ln |z+1| = \ln |x| + C \]

\[ z - C = \ln |x| + \ln |1+z| = \ln |x(1+z)| \]

(exponentiate both sides)

\[ |x(1+z)| = e^{z-C} = e^z e^{-C} \rightarrow x(1+z) = Ae^z \] (A is a \( \pm \) const)

now \( z = \frac{y}{x} \) \[ \Rightarrow \frac{x(1 + \frac{y}{x})}{x} = Ae^y \]

→ can't readily solve for \( y(x) \), but we can at least check our solution in order to verify whether it satisfies the ODE