Let's come back to a recursive sequence we've already defined, the logistic map:

\[ x_{n+1} = \gamma x_n (1-x_n) \quad [\gamma - \text{positive const.}] \]

Going to explore this 'map' in more detail. Consider a starting value \( x_0 \). Then:

\[
\begin{align*}
X_1 &= \gamma x_0 (1-x_0) \\
X_2 &= \gamma x_1 (1-x_1) = \gamma^2 (1-x_0) (1-\gamma x_0 + \gamma x_0^2) \\
X_3 &= \gamma x_2 (1-x_2) = \gamma^3 (1-x_0) x_0 (1-\gamma x_0 + \gamma x_0^2)(1-\gamma^2 x_0 + \gamma^2 x_0^2 + \gamma^3 x_0^3 - 2\gamma^3 x_0^3 + \gamma^3 x_0^4)
\end{align*}
\]

This obviously gets complicated quickly! Clearly if \( |x_0| > 1 \), then the sequence will diverge. If \( x_0 = 1 \), then you essentially have an equilibrium point. Let's simply consider the case \( 0 \leq x_0 \leq 1 \). Then we can examine things visually:

*Graph showing \( x_{n+1} = x_n \)

Limit here spirals into a limit (where the two curves intersect)

*Graph showing \( x_{n+1} = \gamma x_n (1-x_n) \)

(for a particular choice of \( \gamma \))

**Cobweb Plot**: Use the \( x_{n+1} = x_n \) and \( x_{n+1} = \gamma x_n (1-x_n) \) curves to determine the path subsequent iterations will take:

- From \( x_n \), move up to \( x_{n+1} = \gamma x_n (1-x_n) \) curve
- Move left to \( x_{n+1} = x_n \) curve (or to the right)
- Move either up or down back to the \( x_{n+1} = \gamma x_n (1-x_n) \) curve
- Repeat

**Website of interest**: http://math.la.asu.edu/~chaos/logistic.html
You can change your starting value \( x_0 \), but more importantly \( r \) as well.

- Note that if \( r \geq 4 \), things will eventually diverge (consider why!), so we will limit our consideration to \( 0 < r < 4 \).

- Different behaviors are observed for different values of \( r \) (independent of \( x_0 \)):

  - \( 0 < r < 1 \) : \( \lim_{n \to \infty} x_n = 0 \) (things head straight into the limit)
  - \( 1 < r < 2 \) : \( \lim_{n \to \infty} x_n = \frac{r-1}{r} \) (things head straight into the limit)
  - \( 2 < r < 3 \) : ditto, but \( \cdots \) (will spiral in, oscillating about the limit)
  - \( r > 3 \) : complex things start to happen \( \cdots \)

\[ \Rightarrow \text{this is a very useful place to employ a bifurcation diagram} \]