The Approximation Method of Euler

June 9, 2016

In many circumstances, we will not be able to use the techniques of calculus to determine the solution for the initial value problem

\[ y' = f(x, y), \quad y(x_0) = y_0. \]

We consequently rely on numerical techniques to approximate a solution. The first is Euler’s method. This method relies on the tangent line or the first order Taylor’s approximation an differential function.

Begin by choosing a (small) positive number \( h \), then

\[ y(x + h) \approx y(x) + hy'(x) = y(x) + hf(x, y). \]

This tangent line approximation is used iteratively beginning with the point \((x_0, y_0)\)

\[
\begin{align*}
x_1 &= x_0 + h \\
x_2 &= x_1 + h = x_0 + 2h \\
& \vdots \\
x_n+1 &= x_n + h = x_0 + nh \\
y_n+1 &= y_n + hf(x_n, y_n)
\end{align*}
\]

Continue the iteration until some final value for \( x_f \) is obtained. In this case,

\[ n = \frac{x_f - x_0}{h}. \]

Let’s implement the Euler method for the differential equation for the differential equation,

\[ y' = \frac{x}{y}, \quad y(0) = 2 \] (1)

We know that the solution is \( y(x) = \sqrt{x^2 + 4} \).

Take \( h = 0.1 \) and \( x_f = 1 \). So, \( y(1) = \sqrt{5} \). Here is the implementation in R

```r
f <- function(x, y) x/y
x0 <- 0; xf <- 1; y0 <- 2; h <- 0.1
n <- ceiling((xf-x0)/h); x <- numeric(n); y <- numeric(n)
x[1] <- x0; y[1] <- y0
for (i in 2:n){x[i] <- x[i-1] + h; y[i] <- y[i-1] + h*f(x[i-1], y[i-1])}
```

We print the output in a data.frame
> data.frame(x,y,sqrt(x^2+4))
  x     y \sqrt{x^2+4}
1 0.1 2.000000  2.002498
2 0.2 2.005000  2.009975
3 0.3 2.014975  2.022375
4 0.4 2.029864  2.039608
5 0.5 2.049569  2.061553
6 0.6 2.073965  2.088061
7 0.7 2.102895  2.118962
8 0.8 2.136182  2.154066
9 0.9 2.173632  2.193171
10 1.0 2.215038  2.236068

The **relative error** is

\[
\frac{\text{exact} - \text{approximation}}{\text{exact}}.
\]

For this case, we have the table for the relative error.

<table>
<thead>
<tr>
<th>h</th>
<th>y(1)</th>
<th>relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>2.2105</td>
<td>1.14 \times 10^{-2}</td>
</tr>
<tr>
<td>0.01</td>
<td>2.2339</td>
<td>9.49 \times 10^{-4}</td>
</tr>
<tr>
<td>0.001</td>
<td>2.2359</td>
<td>9.29 \times 10^{-5}</td>
</tr>
</tbody>
</table>

**Exercise 1.** Check that \( y(x) = 2e^x - x - 1 \) is an explicit solution to

\[
y' = x + y, \quad y(0) = 2
\]

**Exercise 1.** Compare this to Euler’s method with \( h = 0.1 \) and \( x = 0.1, 0.2, \ldots, 1.0 \).