Cauchy-Euler Equations and Method of Frobenius

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Certain singular equations have a solution that is a series expansion. We begin this investigation with Cauchy-Euler equations.

1 Cauchy-Euler Equations

A second order Cauchy-Euler equation has the form

\[ ax^2 y'' + bxy' + cy = 0 \]  

(1)

for constants \(a\), \(b\), and \(c\). Thus, \(x_0 = 0\) is a singular point.

If we try a solution of the form

\[ y(x) = x^r, \]

then

\[ 0 = ax^2 y'' + bxy' + cy = (ar(r - 1) + br + c)x^r = 0. \]

Thus \(x^r\) is a solution if

\[ ar(r - 1) + br + c = ar^2b - c)r + c = 0 \]

(2)

The equation (2) is called the indicial or characteristic equation.

Example 1. For

\[ x^2 y'' - 4xy' + 6y = 0, \]

the indicial equation is

\[ 0 = r(r - 1) - 4r + 6 = r^2 - 5r + 6 = (r - 2)(r - 3). \]

and

\[ y_1(x) = x^2 \quad \text{and} \quad y_2(x) = x^3 \]

are linearly independent solutions.

As with the constant coefficient equations, we have two additional considerations.

- If the roots in (2) are repeated, i.e., the indicial equation is \(a(r - r_0)^2\), then the two solutions to (1) are

\[ y_1(x) = x^{r_0} \quad \text{and} \quad y_2(x) = x^{r_0} \ln x. \]
• If the roots in (2) are complex conjugates, \( \alpha \pm i\beta \), then the two solutions to (1) are
\[
y_1(x) = x^\alpha \cos(\beta \ln x) \quad \text{and} \quad y_2(x) = x^\alpha \sin(\beta \ln x).
\]

Exercise 2. • Show that
\[
y(x) = c_1 \sqrt{x} + c_2 \sqrt{x} \ln x
\]
\[
4x'y'' + y = 0
\]

• Show that
\[
y(x) = \frac{1}{x} (c_1 \sin(2 \ln x) + c_2 \cos(2 \ln x))
\]
is a general solution to
\[
x^2y'' + 3xy' + 5y = 0
\]

This last equation shows different behavior possible in Cauchy-Euler equations. The solutions are unbounded and oscillate more and more rapidly near \( x = 0 \).

2 Method of Frobenius

We moved from second order constant coefficient ordinary differential equations to differential equations having coefficients that are analytic functions of \( x \). The method of Frobenius makes a similar generalization from the Cauchy-Euler equations.

In this case, we start with
\[
a(x)x^2y'' + b(x)xy' + c(x)y = 0
\]
and divide so that we have
\[
y'' + \frac{b(x)}{xa(x)}y' + \frac{c(x)}{x^2a(x)}y = 0
\]
\[
y'' + p(x)y' + q(x)y = 0
\]

So,
\[
 xp(x) = \frac{b(x)}{a(x)} \quad \text{and} \quad x^2q(x) = \frac{c(x)}{a(x)}.
\]

If we have the limits
\[
\lim_{x \to 0} xp(x) = \lim_{x \to 0} \frac{b(x)}{a(x)} = p_0 \quad \text{and} \quad \lim_{x \to 0} x^2q(x) = \lim_{x \to 0} \frac{c(x)}{a(x)} = q_0.
\]

Then, for \( x \) near zero,
\[
p(x) \approx xp_0 \quad \text{and} \quad q(x) \approx x^2q_0,
\]
the solutions to (3) should be similar to the Cauchy-Euler equation
\[
y'' + \frac{p_0}{x}y' + \frac{q_0}{x^2}y = 0
\]
\[
x^2y'' + xp_0y' + q_0y = 0
\]

(4)
We turn these observations into a definition.

A singular point $x_0$ of the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

is said to be a regular singular point if both

$$(x - x_0)p(x) \quad \text{and} \quad (x - x_0)^2q(x)$$

are analytic at $x_0$. Otherwise $x_0$ is called an irregular singular point.

Returning to (4), we again have an indicial equation

$$r(r-1) + p_0r + q_0 = 0.$$ 

The roots of the indicial equation are called the exponents or indices of the singularity $x_0$.

Example 3. For the Bessel differential equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0, \quad \alpha \geq 0$$

$$y'' + \frac{1}{x}y' + \frac{x^2 - \alpha^2}{x^2}y = 0$$

(5)

To see that $x_0 = 0$ is a regular singular point, note that

$$\lim_{x \to 0} xp(x) = 1 \quad \text{and} \quad \lim_{x \to 0} x^2q(x) = -\alpha^2.$$

The indicial equation is

$$0 = r(r-1) + \alpha^2 = r^2 - \alpha^2$$

So the roots are $\pm \alpha$.

To begin, let’s assume that this equation has distinct real roots, $r_-$ and $r_+$. The method of Frobenius suggests that we look for solutions of the form

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r}$$

For each of the roots to lead to distinct powers, the difference $r_+ - r_-$ cannot be an integer.

We apply this to Bessel’s equation (5) by first writing series expansions for $y$, $y'$ and $y''$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha} = a_0 x^\alpha + a_1 x^{1+\alpha} + a_2 x^{2+\alpha} + a_3 x^{3+\alpha} + a_4 x^{4+\alpha} + \cdots.$$

$$y'(x) = \sum_{n=0}^{\infty} (n+\alpha) a_n x^{n+\alpha-1} = \alpha a_0 x^{\alpha-1} + (1+\alpha)a_1 x^{\alpha} + (2+\alpha)a_2 x^{1+\alpha} + (3+\alpha)a_3 x^{2+\alpha} + \cdots.$$

$$y''(x) = \sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2} = \alpha(\alpha-1)a_0 x^{\alpha-2} + (1+\alpha)\alpha a_1 x^{\alpha-1} + (2+\alpha)(1+\alpha)a_2 x^{\alpha} + \cdots.$$

For Bessel’s equation
\[-\alpha^2 y(x) = -\sum_{n=0}^{\infty} \alpha^2 a_n x^{n+\alpha} = -\alpha^2 a_0 x^\alpha - \alpha^2 a_1 x^{1+\alpha} - \alpha^2 a_2 x^{2+\alpha} - \alpha^2 a_3 x^{3+\alpha} - \alpha^2 a_4 x^{4+\alpha} + \cdots.\]

\[x^2 y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha+2} = a_0 x^{2+\alpha} + a_1 x^{3+\alpha} + a_2 x^{4+\alpha} + a_3 x^{5+\alpha} + \cdots.\]

\[xy(x) = \sum_{n=0}^{\infty} (n + \alpha) a_n x^{n+\alpha} = \alpha a_0 x^\alpha + (1 + \alpha) a_1 x^{1+\alpha} + (2 + \alpha) a_2 x^{2+\alpha} + (3 + \alpha) a_3 x^{3+\alpha} + \cdots.\]

\[x^2 y''(x) = \sum_{n=0}^{\infty} (n + \alpha)(n + \alpha - 1) a_n x^{n+\alpha} = \alpha(\alpha - 1) a_0 x^\alpha + (1 + \alpha) a_1 x^{\alpha+1} + (2 + \alpha) (1 + \alpha) a_2 x^{\alpha+2} + \cdots.\]

To line up the indices, we adjust the expression for the \(x^2 y(x)\) term.

\[x^2 y(x) = \sum_{n=2}^{\infty} a_{n-2} x^{n+\alpha} = a_0 x^{2+\alpha} + a_1 x^{3+\alpha} + a_2 x^{4+\alpha} + a_3 x^{5+\alpha} + \cdots.\]

and add. Notice that the sum on the \(x^\alpha\) term is

\[-\alpha^2 a_0 + \alpha a_0 + \alpha(\alpha - 1) a_0 = 0.\]

So, \(a_0\) is arbitrary. For the \(x^{\alpha+1}\) term

\[-\alpha^2 a_1 + (1 + \alpha) a_1 + (1 + \alpha) \alpha a_1 = 0, \quad (2\alpha + 1) x_1 = 0, \quad \text{and} \quad a_1 = 0.\]

For the powers of \(x^{n+\alpha}\) we have that

\[0 = (n + \alpha)(n + \alpha - 1) a_n + (n + \alpha) a_n - \alpha^2 a_n + a_{n+2}\]

\[= ((n - \alpha)^2 - \alpha^2) a_n + a_{n+2}\]

\[= (n + 2\alpha) a_n + a_{n-2}\]

From this we see that the terms \(a_n\) for \(n\) odd vanish. Also, notice this recursion relation agrees with the case \(\alpha = 0\) determined earlier.

For the even values of \(n\)

\[a_2 = -\frac{1}{2(2+2\alpha)} a_0 = -\frac{1}{2(2+2\alpha)} a_0\]

\[a_4 = -\frac{1}{4(4+2\alpha)} a_2 = -\frac{1}{4(4+2\alpha)2(2+2\alpha)} a_0 = -\frac{1}{2^3(2+\alpha)(1+\alpha)} a_0\]

\[a_6 = -\frac{1}{6(6+2\alpha)} a_2 = -\frac{1}{6(6+2\alpha)4(4+2\alpha)2(2+2\alpha)} a_0 = -\frac{1}{2^5(3\cdot2+1)(3+\alpha)(2+\alpha)(1+\alpha)} a_0\]

\[a_8 = -\frac{2}{8} a_6 = -\frac{1}{4} a_0\]

\[\vdots = \vdots = \vdots = \vdots = \vdots = \vdots = \vdots = \vdots = \vdots = \vdots\]

\[-a_{2k} = \frac{1}{k^2} a_{2(k-1)} = (-1)^k \frac{1}{2^k k!(k+\alpha)_k} a_0\]

The term \((a)_k = a(a-1)\cdots(a-k+1)\) is called the falling factorial and is read “\(x\) falling \(k\)”.

Thus, \(a_0 J_\alpha(x)\) is a solution to (5) where

\[J_\alpha(x) = \sum_{n=0}^{\infty} (-1)^k \frac{1}{2^{2k} k!(k+\alpha)_k} x^{2k} = \sum_{n=0}^{\infty} (-1)^k \frac{1}{k!(k+\alpha)_k} \left(\frac{x}{2}\right)^{2k}\]