[20pts] **Problem 1.**

(a) Mark whether the statement is true or false. (You do not have to provide an explanation)

- If \( f''(x) > 0 \) then \( f(x) \) is increasing. 
  - \( T \) \( f(x) = x^2 \) satisfies \( f'' > 0 \)
  - \( F \) but \( f \) is not

- The fifth derivative of \( 2^x \) is \( 2^x (\ln 2)^5 \).
  - \( T \)

- If \( f(x) \) is continuous on \( (0, 3) \) and \( f(1) = f(2) = 0 \), then \( f(c) = 0 \) for some \( 1 < c < 2 \).
  - \( F \)

- The seventh derivative of \( \sin x \) is \( \cos x \).
  - \( T \)

- If \( f'(x)f''(x) < 0 \) then \( f'(x)^2 \) is decreasing.
  - \( F \)

(b) Given the graph \( y = f(x) \) below, determine the value(s) of \( x \) in the domain of \( f \), at which

- \( f \) has a discontinuity:
  - \( x = -2 \) as \( \lim_{x \to -2} f(x) = -\infty \neq f(-2) \); \( \text{NB} \) \( x = 1 \) is not a point of discontinuity as \( f \) is not defined at \( x = 1 \).
  - \( x = -1 \) as the graph of \( f \) has a cusp (slope \( = \infty \), right slope \( = -\infty \))
  - \( x = 0 \) as the graph has a corner.

- \( f \) is differentiable but not continuous.
  - \( \text{no such } x \) If \( f \) is differentiable at \( x \), it is continuous at \( x \).
**Problem 2.** Given the information about the differentiable functions $f(x)$ and $g(x)$ in the table below, answer the following questions.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
<th>$g(x)$</th>
<th>$f'(x)$</th>
<th>$g'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>-4</td>
<td>-3</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>3</td>
<td>-1</td>
<td>-2</td>
</tr>
</tbody>
</table>

(a) Find the equation of the tangent line to the graph of $f$ at the point $(1, 2)$;

$$y - 2 = f'(1) (x - 1) ~\Rightarrow~ y = 2 + (-3)(x - 1)$$

$$\Rightarrow \quad y = -3x + 5 \quad \text{is the equation of the tangent line at (1, 2)}$$

(b) Compute $h'(2)$ if $h(x) = f(x)/g(x)$;

$$h'(x) = \frac{f'(2)g(2) - f(2)g'(2)}{g(2)^2} = \frac{(-1)3 - 0(-2)}{3^2} = \frac{-4}{9}$$

(c) Compute $h'(1)$ if $h(x) = g(f(x))$;

$$h'(1) = g'(f(1))f'(1) = g'(2)f'(1) = (-2)(-3) = 6$$

(d) Find the equation of the tangent line to the graph $y = h(x)$ at $x = 4$ if $h(x) = g(\sqrt{x}) + (f(x - 3))^2$.

$$h'(x) = g'(\sqrt{x}) \frac{1}{2\sqrt{x}} + 2f(x-3)f'(x-3) (x-3)' = g'(\sqrt{x}) \frac{1}{2\sqrt{x}} + 2f(x-3)f'(x-3)$$

$$\Rightarrow h'(4) = g'(2) \frac{1}{2\sqrt{2}} + 2f(1)f'(1) = \frac{-2}{4} + 2 \cdot 2 \cdot (-3) = -12.5$$

The equation is given by $y - h(4) = h'(4)(x-4)$;

$$\Rightarrow y = g(\sqrt{4}) + f(1)^2 + (-12.5)(x-4) = 3 + 2^2 - 12.5x + 50$$

$$= 57 - 12.5x$$
Problem 3. The graph of \( f'(x) \), the derivative of \( f(x) \), is given below.

(a) On which intervals is \( f \) increasing? Select all that apply.
- A. \((0, x_1)\)
- B. \((x_1, x_2)\)
- C. \((x_2, x_3)\)
- D. \((x_3, x_4)\)
- E. \((x_4, x_5)\)

(b) At which point(s) does \( f \) attain a local maximum? \( f' \) changes sign from + to -
- A. \(x_1\)
- B. \(x_2\)
- C. \(x_3\)
- D. \(x_4\)

(c) On which intervals is \( f \) concave down? Select all that apply.
- A. \((0, x_1)\)
- B. \((x_1, x_2)\)
- C. \((x_2, x_3)\)
- D. \((x_3, x_4)\)
- E. \((x_4, x_5)\)

Problem 4. Let \( Q(T) \) be the amount of ice-cream, in thousands of pints, sold on a single day in Tucson when the average daily temperature is \( T \) degrees Fahrenheit.
(a) Give a practical interpretation of \( Q'(90) = 4 \). Include units.
\[
Q'(T) = \frac{dQ}{dT}, \text{ units = thousands of pints} \quad \text{; } Q'(90) = 4 \text{ means that when the temperature is 90°F, about 4000 pints of ice-cream will be sold if the daily temperature increases by 1°F.}
\]
(b) Assuming that \( Q(T) \) is an increasing function of \( T \), give a practical interpretation of the statements \( Q^{-1}(92.3) = 90 \) and \([Q^{-1}]'(92.3) = 0.25\).
\[
Q^{-1}(92.3) = 90 \text{ means that on a day when 92,300 pints of ice-cream are sold, the temperature is 90°F. } (Q^{-1})' = \frac{dT}{dQ} = \text{units = °F/thousands of pints} \quad \text{; } Q^{-1}(92.3) = 0.25 \text{ means that when 92,300 pints are sold, it will take a daily temp increase of 0.25°F for an additional 1000 pints to be sold.}
\]
(c) Estimate the average temperature on the day when 89,900 pints of ice-cream were sold.
\[
\text{Want to estimate } Q^{-1}(89.9) = Q^{-1}(92.3 - 2.4) \approx Q^{-1}(92.3) - 2.4(Q^{-1})'(92.3) = 90 - (2.4)(0.25) = 90 - 0.6 = 89.4°F
\]
Problem 5. Let \( a > 1 \). Show that the polynomial \( p(x) = ax^4 - 2ax + a - 1 \) has a root in \([1, 1]\).

A polynomial is a continuous function on \((\infty, \infty)\). In particular, \( p(x) \) is continuous on \([-1, 1]\). Also,

\[
p(-1) = a + 2a + a - 1 = 4a - 1 > 3 \quad \text{for} \quad a > 1; \\
p(1) = a - 2a + a - 1 = -1
\]

Since \( p(-1) < 0 < p(1) \), the Intermediate Value Theorem tells us that \( p(x) = 0 \) for some \( x_0 \) in \([-1, 1]\).

Problem 6. Compute the limits if they exist. If not, argue why.

(a) \[
\lim_{x \to \infty} \frac{x^3 + 2x}{2x^2 - 3x^3 + 1} = \lim_{x \to \infty} \frac{x^3(1 + \frac{2}{x^3})}{x^3 \left( \frac{2}{x} - 3 + \frac{1}{x^3} \right)} = \lim_{x \to \infty} \frac{1 + \frac{2}{x^3}}{\frac{2}{x} - 3 + \frac{1}{x^3}} = \frac{1 + 0}{0 - 3 + 0} = \boxed{-\frac{1}{3}}
\]

(b) \[
\lim_{x \to 2} \frac{x^2|x - 2|}{x - 2}
\]

The left limit \( \lim_{x \to 2^-} \frac{x^2|x - 2|}{x - 2} = \lim_{x \to 2^-} \frac{x^2(2-x)}{x - 2} = 4 \lim_{x \to 2^-} x^2 = 4 \)

The right limit \( \lim_{x \to 2^+} \frac{x^2|x - 2|}{x - 2} = \lim_{x \to 2^+} \frac{x^2(2-x)}{x - 2} = 4 \lim_{x \to 2^+} x^2 = 4 \)

The left limit \( \neq \) right limit, so the limit does not exist.

(c) \[
\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1} (x+1) = 2
\]