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Model ingredients:
- The flow or *flux* is a vector field $\mathbf{J}(x, t)$, so that $\mathbf{J} \cdot \hat{n} \, dA$ is flow across infinitesimal area $\hat{n} \, dA$.
- $Q(x, t)$ is the rate of inflow at point $x$. 

Conservation of $u(x, t)$ on any region $R \subset \Omega$ implies

$$
\frac{d}{dt} \int_{R} u \, dx = \int_{R} \frac{\partial u}{\partial t} \, dx = -\int_{\partial R} J \cdot \hat{n} \, dx + \int_{R} Q(x, t) \, dx.
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(recall notation: $\partial R$ is boundary of $R$, $\hat{n}$ is outward normal vector)
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Use divergence theorem to turn boundary integral into integral on region $R$,

$$\int_R \left( \frac{\partial u}{\partial t} + \nabla \cdot \mathbf{J} - Q \right) \, dx = 0.$$
Deriving a PDE for a conserved quantity

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Since this is true for any subregion \( R \), integrand is zero:

\[
\frac{\partial u}{\partial t} + \nabla \cdot \mathbf{J} = Q.
\]

conservation form/continuity equation/transport equation
Flux-type boundary conditions specify flow $F(x) : \partial\Omega \to \mathbb{R}$ through physical boundary.
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$$\mathbf{J}(u, \nabla u, \ldots) \cdot \hat{n} = F(\mathbf{x}), \quad \mathbf{x} \in \partial \Omega.$$
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More specifically, if boundary is insulating, get "Neumann" boundary condition

$$\nabla u \cdot \hat{n} = 0.$$ 

Remark: Dirichlet boundary condition $u = U(x), \quad x \in \partial \Omega$ will not guarantee flux is zero at boundary.
Suppose that $u = u(x, t)$ is transported at velocity $c$, so that (one dimensional) scalar flux is $J = cu$.
Example: transport and traffic flow

Suppose that \( u = u(x, t) \) is transported at velocity \( c \), so that (one dimensional) scalar flux is \( J = cu \).

Then \( \frac{\partial u}{\partial t} + \nabla \cdot J = Q \) becomes (assuming \( Q = 0 \))

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Traffic flow: speed can be modeled as a decreasing function of density \( c = c_0 - mu \), so \( J = u(c_0 - mu) \); conservation law becomes

\[
    u_t + c_0 u_x - m(u^2)_x = 0. \quad \text{(nonlinear transport equation)}
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Example: Diffusion with a source

Random motions of particles (and other things) leads to diffusion. Means that net flow has a direction toward regions of less density.
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Simplest model: Fick/Fourier law $\mathbf{J} = -D \nabla u$. Sources $Q(x, t)$ created by, for example heat production or chemical reactions. The conservation equation becomes

$$u_t = D \nabla \cdot \nabla u + Q = D \Delta u + Q,$$

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Momentum conservation leads to

$$(u_t)_t = c^2 \nabla \cdot \nabla u = c^2 \Delta u,$$  \hspace{1cm} \text{(wave equation)}$$

where $c^2 = \sigma / \rho$. 
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Steady state equations

Often dynamical processes “settle down”; for PDEs this means the time derivative can be ignored. For a conservation law with flux $J(u)$ and time independent source term $Q$, a *steady state solution* solves

$$\nabla \cdot J(u) = Q.$$ 

Interpretation: the amount flowing into a region in space equals the amount flowing out.
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Example (diffusion with a source): Flux is given by Fick’s law $J = -D\nabla u$, and $Q(x, y)$ is a prescribed source term. Steady state $u = u(x, y)$ solves

$$D\nabla \cdot \nabla u = \Delta u = Q(x, y).$$

If $Q \neq 0$, get *Poisson’s equation*; if $Q \equiv 0$, get *Laplace’s equation*. 
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- Even if we could, often hard to see the essential aspects.

Idea: use coarse-grained quantities to study solutions qualitatively. Some of these are inspired by physics (energy, entropy), whereas others are completely abstract.
A *functional* $F[u]$ maps $u$ to the real numbers, e.g.

$$F[u] = \int_{\Omega} u(x) \, dx \quad \text{or} \quad F[u] = \int_{\Omega} |\nabla u|^2 \, dx.$$ 

In our case, these are often quantities of physical interest (mass, energy, momentum)
Conserved and dissipated quantities

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Let $u(x, t) : \Omega \times [0, \infty) \to \mathbb{R}$ be a solution of some PDE, and suppose $F[u]$ has the form

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Time evolution of $F[u]$ may be categorized as:

- If $dF/dt = 0$ for all $u$, then $F$ is called conserved,
- If $dF/dt \leq 0$ for all $u$, then $F$ is called dissipated.
Example: energy in the wave equation

Let $u$ solve

$$u_{tt} = u_{xx}, \quad u(0, t) = 0 = u(L, t).$$
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$$\frac{dE}{dt} = \int_0^L u_t u_{tt} + u_x u_{xt} \, dx,$$

If $u(x, 0) = 0 = u_t(x, 0)$ initially, does the solution remain zero?

Yes, since $E(0) = 0$, $E(t) \equiv 0$, thus $u_x \equiv 0$. Using boundary conditions gives $u(x, t) \equiv 0$.

Converse also true: if $u(x, 0) \neq 0$ initially, then solution never "dies out".
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$$\frac{dE}{dt} = u_x u_t \bigg|_{x=0}^{x=L} + \int_{0}^{L} u_t u_{tt} - u_{xx} u_t \, dx,$$

Then use equation and boundary conditions to get $dE/dt = 0$. If $u(x, 0) = 0 = u_t(x, 0)$ initially, does the solution remain zero? Yes, since $E(0) = 0$, $E(t) \equiv 0$, thus $u_x \equiv 0$. Using boundary conditions gives $u(x, t) \equiv 0$. Conversely, if $u(x, 0) \neq 0$ initially, then solution never "dies out."
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is dissipated,

One interpretation: arclength of \( x \)-cross sections of \( u \) can be approximated

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    \int_{0}^{L} \sqrt{1 + u_x^2} \, dx \approx \int_{0}^{L} 1 + \frac{1}{2} u_x^2 \, dx.
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Since \( dF/dt \leq 0 \), arclength diminishes and spatial oscillations die away.
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