Solutions of differential equations using transforms

Process:

- Take transform of equation and boundary/initial conditions in one variable.
- Derivatives are turned into multiplication operators.
- Solve (hopefully easier) problem in $k$ variable.
- Inverse transform to recover solution, often as a convolution integral.
Ordinary differential equations: example 1

\[- u'' + u = f(x), \quad \lim_{|x| \to \infty} u(x) = 0.\]
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Transform using the derivative rule, giving

\[k^2 \hat{u}(k) + \hat{u}(k) = \hat{f}(k).\]

Just an algebraic equation, whose solution is

\[\hat{u}(k) = \frac{\hat{f}(k)}{1 + k^2}.\]
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Inverse transform of product of \(\hat{f}(k)\) and \(1/(1 + k^2)\) is convolution:

\[u(x) = f(x) \ast \left( \frac{1}{1 + k^2} \right) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-y|} f(y) dy.\]

But where was far field condition used?
Example 2. The *Airy* equation is

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\[ \hat{u}(k) = Ce^{ik^3/3}. \]
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Inverse transform is

\[ u(x) = \frac{C}{2\pi} \int_{-\infty}^{\infty} \exp(i[kx + k^3/3]) \, dk. \]

With the choice \( C = 1 \) get the *Airy function*. 
Partial differential equations, example 1

Laplace equation in upper half plane:

\[ u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y > 0, \]

\[ u(x, 0) = g(x), \quad \lim_{y \to \infty} u(x, y) = 0. \]
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Transform in the \( x \) variable only:

\[ U(k, y) = \int_{-\infty}^{\infty} e^{-ikx} u(x, y) \, dx. \]

Note \( y \)-derivatives commute with the Fourier transform in \( x \).

\[ -k^2 U + U_{yy} = 0, \quad U(k, 0) = \hat{g}(k), \quad \lim_{y \to \infty} U(k, y) = 0. \]
Now solve ODEs

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General solution is \(U = c_1(k)e^{+|k|y} + c_2(k)e^{-|k|y}.\) Using boundary conditions,

\[U(k, y) = \hat{g}(k)e^{-|k|y}.\]
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Inverse transform using convolution and exponential formulas

\[u(x, y) = g(x) \ast \left( e^{-|k|y} \right) \vee = g(x) \ast \left( \frac{y}{\pi(x^2 + y^2)} \right)\]

\[= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yg(x_0)}{(x - x_0)^2 + y^2} dx_0.\]

Same formula as obtained by Green’s function methods!
“Transport equation"

\[ u_t + cu_x = 0, \quad -\infty < x < \infty, \quad t > 0, \quad u(x,0) = f(x). \]
Partial differential equations, example 2

“Transport equation"

\[ u_t + cu_x = 0, \quad -\infty < x < \infty, \quad t > 0, \quad u(x, 0) = f(x). \]

As before,

\[ U(k, t) = \int_{-\infty}^{\infty} e^{-ikx} u(x, t) dx. \]

therefore transform in \( x \) variables is

\[ U_t + ikcU = 0, \quad U(k, 0) = \hat{f}(k). \]
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Simple differential equation with solution

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Simple differential equation with solution

\[ U(k, t) = e^{-ickt}\hat{f}(k). \]

Use translation formula \( f(x - a) = e^{-iat}\hat{f}(k) \) with \( a = ct \),

\[ u(x, t) = f(x - ct). \]
Consider the wave equation on the real line

\[ u_{tt} = u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \]
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Solution of initial value problem

\[ U(k, t) = \hat{f}(k) \cos(kt) + \frac{\hat{g}(k)}{k} \sin(kt). \]

Need inverse transform formulas for \( \cos, \sin \) and \( \hat{f}/k \).
For $\cos, \sin$, write in terms of complex exponentials:

$$[\cos(ak)]^\vee = \frac{1}{2}[e^{iak} + e^{-iak}]^\vee = \frac{1}{2}[\delta(x + a) + \delta(x - a)],$$

$$[\sin(ak)]^\vee = -\frac{i}{2}[e^{iak} - e^{-iak}]^\vee = -\frac{i}{2}[\delta(x + a) - \delta(x - a)].$$
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\]

If \(f(x)\) is integrable, int. by parts gives

\[
\left[\int_{-\infty}^{x} f(x')dx'\right]^\wedge = \int_{-\infty}^{\infty} e^{-ikx} \int_{-\infty}^{x} f(x')dx' dx = \frac{1}{ik} \int_{-\infty}^{\infty} e^{-ikx} f(x)dx.
\]
For \( \cos, \sin \), write in terms of complex exponentials:

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If \( f(x) \) is integrable, int. by parts gives

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\]

Therefore

\[
\left[ \frac{\hat{f}(k)}{ik} \right]^\vee = \int_{-\infty}^{x} f(x') dx'.
\]

That is, integration in \( x \) gives a factor of \( 1/(ik) \) in the transform.
Now for inverse transform of

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First term:

\[
\left[ \hat{f}(k) \cos(kt) \right] \hat{\gamma} = f(x) \ast \frac{1}{2} [\delta(x + t) + \delta(x - t)]
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} f(x - y)[\delta(y + t) + \delta(y - t)]dy = \frac{1}{2}[f(x - t) + f(x + t)].
\]
Now for inverse transform of

\[ U(k, t) = \hat{f}(k) \cos(kt) + \frac{\hat{g}(k)}{k} \sin(kt). \]

First term:

\[ \left[ \hat{f}(k) \cos(kt) \right]^{\neg} = f(x) \ast \frac{1}{2} \left[ \delta(x + t) + \delta(x - t) \right] \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} f(x - y) \left[ \delta(y + t) + \delta(y - t) \right] dy = \frac{1}{2} [f(x - t) + f(x + t)]. \]

Second term:

\[ \left[ \frac{\hat{g}(k) \sin(kt)}{k} \right]^{\neg} = i \int_{-\infty}^{\infty} \left[ \frac{\hat{g}(k) \sin(kt)}{k} \right]^{\neg} (x') dx' \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} g(x') \ast \left[ \delta(x' + t) - \delta(x' - t) \right] dx' \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x' - y) \left[ \delta(y + t) - \delta(y - t) \right] dy dx' \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} g(x' + t) - g(x' - t) dx'. \]
Finally, can change variables $\xi = x' + t$ or $\xi = x' - t$

$$\int_{-\infty}^{x} g(x' + t) - g(x' - t)dx' = \int_{-\infty}^{x+t} g(\xi)d\xi - \int_{-\infty}^{x-t} g(\xi)d\xi$$

$$= \int_{x-t}^{x+t} g(\xi)d\xi.$$
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$$= \int_{x-t}^{x+t} g(\xi)d\xi.$$ 

All together get *d’Alembert’s* formula

$$u(x, t) = \frac{1}{2}[f(x - t) + f(x + t)] + \frac{1}{2} \int_{x-t}^{x+t} g(\xi)d\xi.$$
Consider generic, linear, time-dependent equation

\[ u_t(x, t) = \mathcal{L}u(x, t), \quad -\infty < x < \infty, \quad u(x, 0) = f(x), \quad \lim_{|x| \to \infty} u(x, t) = 0, \]

where \( \mathcal{L} \) is some operator (e.g. \( \mathcal{L} = \partial^2 / \partial x^2 \)).
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where $\mathcal{L}$ is some operator (e.g. $\mathcal{L} = \partial^2/\partial x^2$).

The **fundamental solution** $S(x, x_0, t)$ is a type of Green’s function, solving

$$S_t = \mathcal{L}_x S, \quad -\infty < x < \infty, \quad S(x, x_0, 0) = \delta(x - x_0), \quad \lim_{|x| \to \infty} S(x, x_0, t) = 0.$$
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Initial condition means \( S \) limits to a \( \delta \)-function as \( t \to 0 \):

\[ \lim_{t \to 0} \int_{-\infty}^{\infty} S(x, x_0, t)\phi(x)dx = \int_{-\infty}^{\infty} \delta(x - x_0)\phi(x)dx = \phi(x_0), \]
Claim that the initial value problem has solution

\[ u(x, t) = \int_{-\infty}^{\infty} S(x, x_0, t)f(x_0)dx_0, \]
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Check:

\[ u(x, 0) = \lim_{t \to 0} \int_{-\infty}^{\infty} S(x, x_0, t)f(x_0)dx_0 = \int_{-\infty}^{\infty} \delta(x-x_0)f(x_0)dx_0 = f(x). \]
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Plugging \( u \) into the equation and moving time derivative inside the integral

\[ u_t = \int_{-\infty}^{\infty} S_t(x, x_0, t)f(x_0)dx_0 = \int_{-\infty}^{\infty} \mathcal{L}_x S(x, x_0, t)f(x_0)dx_0. \]
Claim that the initial value problem has solution

\[ u(x, t) = \int_{-\infty}^{\infty} S(x, x_0, t)f(x_0)\,dx_0, \]

Check:

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Plugging \( u \) into the equation and moving time derivative inside the integral

\[ u_t = \int_{-\infty}^{\infty} S_t(x, x_0, t)f(x_0)\,dx_0 = \int_{-\infty}^{\infty} \mathcal{L}_x S(x, x_0, t)f(x_0)\,dx_0. \]

Now move operator outside integral

\[ u_t = \mathcal{L}_x \int_{-\infty}^{\infty} S(x, x_0, t)f(x_0)\,dx_0 = \mathcal{L}_x u. \]
For diffusion equation on the real line, $S$ solves

$$S_t = DS_{xx}, \quad -\infty < x < \infty, \quad S(x, x_0, 0) = \delta(x - x_0), \quad \lim_{|x| \to \infty} S(x, x_0, t) = 0.$$
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$$S_t = DS_{xx}, \quad -\infty < x < \infty, \quad S(x, x_0, 0) = \delta(x-x_0), \quad \lim_{|x| \to \infty} S(x, x_0, t) = 0.$$ 

Take Fourier transform in $x$ by letting

$$\hat{S}(k, x_0, t) = \int_{-\infty}^{\infty} S(x, x_0, t)e^{-ikx} \, dx,$$

giving

$$\hat{S}(k, x_0, t) = e^{-ikx_0} - Dk^2 t.$$
For diffusion equation on the real line, $S$ solves

$$S_t = DS_{xx}, \ -\infty < x < \infty, \ S(x, x_0, 0) = \delta(x-x_0), \ \lim_{|x|\to\infty} S(x, x_0, t) = 0.$$ 

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$$\hat{S}_t = -Dk^2 \hat{S}, \ \hat{S}(k, 0) = e^{-ix_0k}.$$
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$$S_t = DS_{xx}, \quad -\infty < x < \infty, \quad S(x, x_0, 0) = \delta(x-x_0), \quad \lim_{|x| \to \infty} S(x, x_0, t) = 0.$$ 

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Solution to this ODE

$$\hat{S} = e^{-ix_0k-Dk^2t}.$$
Inverse transform of

\[ \hat{S} = e^{-ix_0k-Dk^2t}. \]

uses translation, dilation, and Gaussian formulas:

\[
S(x, x_0, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-x_0)^2/(4Dt)}.
\]
Inverse transform of
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uses translation, dilation, and Gaussian formulas:
\[ S(x, x_0, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x-x_0)^2/(4Dt)}. \]
It follows that the solution to \( u_t = Du_{xx} \) and \( u(x, 0) = f(x) \) is
\[ u(x, t) = \int_{-\infty}^{\infty} \frac{f(x_0)}{\sqrt{4\pi Dt}} e^{-(x-x_0)^2/(4Dt)} dx_0. \]
Fundamental solutions using the Fourier transform, example 2

Linearized Korteweg - de Vries (KdV) equation:

\[ u_t = -u_{xxx}, \quad u(x, 0) = f(x), \quad \lim_{|x| \to \infty} u(x, t) = 0. \]
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Transforming

\[ \hat{S}_t = ik^3 \hat{S}, \quad \hat{S}(k, 0) = e^{-ix_0k}, \]

whose solution is \( \hat{S}(k, x_0, t) = e^{-ix_0k} e^{ik^3t} \).
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whose solution is \( \hat{S}(k, x_0, t) = e^{-ix_0 k} e^{ik^3 t} \).

Recall transform of Airy function \( \text{Ai}(x) \) is \( e^{ik^3/3} \), therefore

\[ S(x, x_0, t) = \left[ e^{-ix_0 k} e^{ik^3 t} \right]^\vee = \left[ e^{i(k/a)^3/3} \right]^\vee (x - x_0) \]

\[ = a \text{Ai}\left(a(x - x_0)\right), \quad a \equiv (3t)^{-1/3}. \]
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Recall transform of Airy function \( \text{Ai}(x) \) is \( e^{ik^3/3} \), therefore

\[ S(x, x_0, t) = \left[ e^{-ix_0 k} e^{ik^3 t} \right]^\vee = \left[ e^{i(k/a)^3/3} \right]^\vee (x - x_0) \]

\[ = a \text{Ai} \left( a(x - x_0) \right), \quad a \equiv (3t)^{-1/3}. \]

Solution to original equation:

\[ u(x, t) = \frac{1}{(3t)^{1/3}} \int_{-\infty}^{\infty} \text{Ai} \left( \frac{x - x_0}{(3t)^{1/3}} \right) f(x_0) dx_0. \]
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Fundamental solution must satisfy boundary condition at $x = 0$.
The method of images for fundamental solutions

- For solutions on half-line $x > 0$, can’t use Fourier transform directly.
- Fundamental solution must satisfy boundary condition at $x = 0$
- Inspiration: method of images. If $S_\infty(x; x_0, t)$ is the fundamental solution for the whole line, then:
  - Odd reflection $S = S_\infty(x; x_0, t) - S_\infty(x; -x_0, t)$ gives $S(0, x_0, t) = 0$.
  - Even reflection $S = S_\infty(x; x_0, t) + S_\infty(x; -x_0, t)$ gives $S_x(0, x_0, t) = 0$. 


The method of images for fundamental solutions, example

Consider diffusion equation on half line:

\[ u_t = Du_{xx}, \quad u(x, 0) = f(x), \quad u(0, t) = 0, \quad \lim_{x \to \infty} u(x, t) = 0. \]
Consider diffusion equation on half line:

\[ u_t = Du_{xx}, \quad u(x, 0) = f(x), \quad u(0, t) = 0, \quad \lim_{x \to \infty} u(x, t) = 0. \]

Use odd reflection of fundamental solution for whole line

\[ S_\infty = e^{-(x-x_0)^2/(4Dt)} / \sqrt{4\pi Dt}, \]

\[ S(x, x_0, t) = \frac{1}{\sqrt{4\pi Dt}} \left[ e^{-(x-x_0)^2/(4Dt)} - e^{-(x+x_0)^2/(4Dt)} \right] \]
Consider diffusion equation on half line:

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Use odd reflection of fundamental solution for whole line
\[ S_\infty = e^{-(x-x_0)^2/(4Dt)}/\sqrt{4\pi Dt}, \]
\[ S(x, x_0, t) = \frac{1}{\sqrt{4\pi Dt}} \left[ e^{-(x-x_0)^2/(4Dt)} - e^{-(x+x_0)^2/(4Dt)} \right] \]

Therefore the solution \( u \) is just
\[ u(x, t) = \int_0^\infty \frac{f(x_0)}{\sqrt{4\pi Dt}} \left[ e^{-(x-x_0)^2/(4Dt)} - e^{-(x+x_0)^2/(4Dt)} \right] dx_0. \]
The age of the earth

Lord Kelvin: simple model of temperature of earth

\[ u(x, t) \] at depth \( x \) and time \( t \)

\[ u_t = Du_{xx}, \quad x > 0, \quad u(x, 0) = U_0, \quad u(0, t) = 0. \]

Scale chosen so \( u = 0 \) on surface; assumes initially constant temperature \( (U_0) \) throughout the molten earth.
Lord Kelvin: simple model of temperature of earth $u(x, t)$ at depth $x$ and time $t$

$$u_t = Du_{xx}, \ x > 0, \ u(x, 0) = U_0, \ u(0, t) = 0.$$ 

Scale chosen so $u = 0$ on surface; assumes initially constant temperature ($U_0$) throughout the molten earth.

We found solution

$$u(x, t) = \int_0^\infty \frac{f(x_0)}{\sqrt{4\pi Dt}} \left[ e^{-(x-x_0)^2/(4Dt)} - e^{-(x+x_0)^2/(4Dt)} \right] dx_0.$$ 

Temperature gradient $\mu$ at surface is therefore

$$\mu = u_x(0, t) = U_0 \sqrt{\frac{1}{\pi Dt}} \int_0^\infty x_0 e^{-x_0^2/(4Dt)} dx_0 = U_0 \sqrt{\frac{1}{\pi Dt}}.$$ 

This relates the age of earth $t$ to quantities we can estimate $U_0 \approx$ melting temp. of iron $\approx 10^4$ C, $D \approx 10^{-3}$ m$^2$/s, $\mu \approx 10^{-2}$ C/m, which gives $t \approx 3 \times 10^7$ years !!?
The age of the earth

Lord Kelvin: simple model of temperature of earth

\[ u(x, t) \text{ at depth } x \text{ and time } t \]

\[ u_t = Du_{xx}, \quad x > 0, \quad u(x, 0) = U_0, \quad u(0, t) = 0. \]

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We found solution

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\]

Temperature gradient \( \mu \) at surface is therefore

\[
 \mu = u_x(0, t) = \frac{U_0}{\sqrt{4\pi Dt}} \frac{1}{Dt} \int_0^\infty x_0 e^{-x_0^2/(4Dt)} dx_0 = \frac{U_0}{\sqrt{\pi Dt}}.
\]

This relates the age of earth \( t \) to quantities we can estimate

\( U_0 \approx \) melting temp. of iron \( \approx 10^4 \) C,

\( D \approx 10^{-3} \) m\(^2\)/s,

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The age of the earth

Lord Kelvin: simple model of temperature of earth \( u(x, t) \) at depth \( x \) and time \( t \)

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The age of the earth

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