Want $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ which solves

$$\Delta u = f(x), \quad \lim_{|x| \rightarrow \infty} u(x) = 0.$$
Higher dimensional Green’s functions

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\]

Green’s function formally solves \( \Delta_x G(x; x_0) = \delta(x - x_0) \), which is same as

\[
\Delta_x G(x; x_0) = 0, \quad x \neq x_0, \quad \lim_{|x| \to \infty} G(x, x_0) = 0.
\]

with “normalization” condition

\[
\int_{\partial B} \nabla_x G(x; x_0) \cdot \hat{n} dx = 1.
\]

where \( B \) is any disk with center \( x_0 \).
Green’s function in 3D, cont.

Symmetry allows $G = G(r)$, $r = |\mathbf{x} - \mathbf{x}_0|$, so that

$$\frac{1}{r^2} (r^2 G'(r))' = 0 \text{ if } r \neq 0, \quad \lim_{r \to \infty} G(r) = 0.$$
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Integrating twice,

$$G = -\frac{c_1}{r} + c_2, \quad c_2 = 0 \text{ by far-field condition}$$
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Integrating twice,
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G = -\frac{c_1}{r} + c_2, \quad c_2 = 0 \text{ by far-field condition}
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Normalization: let $B$ be the unit sphere centered at $x_0$,
\[
1 = \int_{\partial B} \nabla_x G(\mathbf{x}; \mathbf{x}_0) \cdot \hat{n} \, d\mathbf{x} = \int_{\partial B} \frac{c_1}{r^2} \, d\mathbf{x} = 4\pi c_1,
\]
so that $c_1 = 1/4\pi$. 
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so that \( c_1 = 1/4\pi. \)

The Green’s function is therefore \( G(\mathbf{x}; \mathbf{x}_0) = -1/(4\pi |\mathbf{x} - \mathbf{x}_0|) \)
and
\[
\mathbf{u}(\mathbf{x}) = -\int_{\mathbb{R}^3} \frac{f(\mathbf{x}_0)}{4\pi |\mathbf{x} - \mathbf{x}_0|} \, d\mathbf{x}_0^3.
\]
Example: $L = \Delta$ (two dimensions)

$$\Delta u = f, \quad \lim_{r \to \infty} \left( u(r, \theta) - u_r(r, \theta)r \ln r \right) = 0.$$

Look for a Green's function of form

$$G = G(|x - x_0|) = G(r),$$

where

$$\frac{1}{r} \left( rG'(r) \right)' = 0 \text{ if } r \neq 0,$$

$$\lim_{r \to \infty} \left( G - G_{x_0}r \ln r \right) = 0.$$

The general solution is

$$G = c_1 \ln r + c_2,$$

where $c_2 = 0$ by the far-field condition.

Normalization condition (using $B$ = unit disk)

$$1 = \int_{\partial B} \nabla_x G(x; 0) \cdot \hat{n} \, dx = \int_{\partial B} c_1 \frac{|x_0|}{dx} \, dx = 2\pi c_1,$$

so that $c_1 = 1/2\pi$. Thus the Green's function is

$$G(x; x_0) = \frac{\ln |x - x_0|}{2\pi},$$

and

$$u(x) = \int_{B_2} \ln |x - x_0| f(x_0) \, dx_0.$$
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Look for a Green’s function of form \( G = G(|\mathbf{x} - \mathbf{x}_0|) = G(r), \)

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\frac{1}{r} (rG'(r))' = 0 \text{ if } r \neq 0, \quad \lim_{r \to \infty} \left( G - G_r r \ln r \right) = 0.
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The general solution is

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Normalization condition (using \( B = \) unit disk)

\[
1 = \int_{\partial B} \nabla_x G(\mathbf{x}; 0) \cdot \hat{n} \, d\mathbf{x} = \int_{\partial B} \frac{c_1}{|\mathbf{x}_0|} \, d\mathbf{x} = 2\pi c_1,
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so that \( c_1 = 1/2\pi \). Thus the Green’s function is

\[
G(\mathbf{x}; \mathbf{x}_0) = \ln \frac{|\mathbf{x} - \mathbf{x}_0|}{2\pi}, \quad \text{and}
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\[
u(\mathbf{x}) = \int_{\mathbb{R}^2} \frac{\ln |\mathbf{x} - \mathbf{x}_0| f(\mathbf{x}_0)}{2\pi} \, d\mathbf{x}_0^2.
\]
$\Delta u - u = f(x), \quad \lim_{r \to \infty} u = 0, \quad u : \mathbb{R}^2 \to \mathbb{R}.$
\[ \Delta u - u = f(x), \quad \lim_{r \to \infty} u = 0, \quad u : \mathbb{R}^2 \to \mathbb{R}. \]

As before, suppose \( G(x, x_0) = g(r) \) where \( r = |x - x_0| \), so that \( \Delta G - G = \delta(x - x_0) \) is
\[ g'' + \frac{1}{r} g' - g = 0, \quad r \neq 0, \quad \lim_{r \to \infty} g(r) = 0, \]
which is the “modified” Bessel equation of order zero.
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which is the “modified” Bessel equation of order zero. Only linearly independent solution which decays at infinity:

\[
K_0(r) = \int_0^\infty \frac{\cos(rt)}{\sqrt{t^2 + 1}} dt.
\]
Example: Helmholtz operator

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The Green's function is therefore \( G(x, x_0) = cK_0(|x - x_0|) \) where \( c \) is found from a normalization condition

\[ 1 = \int_{\partial B_r(x_0)} \frac{\partial_x G(x, x_0)}{\partial n}(x, x_0) \, dx - \int_{B_r(x_0)} G(x, x_0) \, dx \sim \int_{\partial B_r(x_0)} \frac{\partial_x G(x, x_0)}{\partial n}(x, x_0) \, dx, \]

as \( r \to 0 \).
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as \( r \to 0 \). It can be shown that \( K_0 \sim -\ln(r) \) when \( r \) is small, and therefore

\[ \int_{\partial B_r(x_0)} \frac{\partial x G}{\partial n} (x, x_0) \, dx \sim -c \int_{\partial B_r(x_0)} \frac{1}{r} \, dx = -2\pi c. \]

so that \( c = -1/2\pi \).
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\]
so that \( c = -1/2\pi \). Therefore \( u(x) = -\int_{\mathbb{R}^2} \frac{K_0(|x-x_0|)f(x_0)}{2\pi} dx_0 \).