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\[ u_{tt} = \Delta u \quad (\text{Wave}) \]
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Domain: \((x, y) \in D\), where \(D\) is open set with smooth boundary \(t > 0\) (diffusion/wave) or \(a < z < b\) (Laplace).
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Will need homogeneous boundary conditions such as

\[ u(x, y, \cdot) = 0, \quad (x, y) \in \partial D \quad (\text{Dirichlet}) \]
\[ \nabla u(x, y, \cdot) \cdot \hat{n} = 0, \quad (x, y) \in \partial D \quad (\text{Neumann}) \]

On the other hand, conditions at \( t = 0 \) or \( z = a, b \) are arbitrary.
Separating variables

Look for solutions of form \( u = T(t)v(x, y) \) or \( u = Z(z)v(x, y) \)

\[
\frac{T'}{T} = \frac{\Delta v}{v} = -\lambda \quad (\text{Diffusion})
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\[
\frac{T''}{T} = \frac{\Delta v}{v} = -\lambda \quad (\text{Wave})
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Resulting multidimensional eigenvalue problem: find \( v : D \to \mathbb{R} \)

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\Delta v + \lambda v = 0, \quad \text{plus boundary conditions.}
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For time being, suppose we already know the eigenfunctions \( v_n(x,y) \) and corresponding eigenvalues \( \lambda_n, n = 1, 2, 3, \ldots \).
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With suitable boundary conditions

- eigenvalues are real, non-negative
- Eigenfunctions are orthogonal w.r.t. inner product \( \langle u, v \rangle = \int_D uv \, dx \).
Solution in terms of eigenfunctions and eigenvalues

Solving the ODEs for the $T$ and $Z$ variables and taking a superposition, we arrive at the general solutions

$$u(x, y, t) = \sum_{n=1}^{\infty} A_n \exp(-\lambda_n t)v_n(x, y) \quad (Diffusion)$$

$$u(x, y, t) = \sum_{n=1}^{\infty} [A_n \cos(\sqrt{\lambda_n} t) + B_n \sin(\sqrt{\lambda_n} t)]v_n(x, y) \quad (Wave)$$

$$u(x, y, z) = \sum_{n=1}^{\infty} [A_n \exp(\sqrt{\lambda_n} z) + B_n \exp(-\sqrt{\lambda_n} z)]v_n(x, y) \quad (Laplace)$$
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- Difficult to write complete solution for arbitrary domain $D$. 

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\end{align*}
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- Main issue: solve the eigenvalue problem.
- Difficult to write complete solution for arbitrary domain $D$.
- Three tractable cases are where $D$ is a rectangle, a disk, and the surface of a sphere.
Let \( u, v : D \to \mathbb{R} \) be smooth functions. Apply the divergence theorem to \( u \nabla v \),

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\int_D \nabla \cdot (u \nabla v) \, dx = \int_{\partial D} u \nabla v \cdot \hat{n} \, dx.
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Use $\nabla \cdot (u \nabla v) = \nabla u \cdot \nabla v + u \Delta v$,

$$\int_D u \Delta v \, d\mathbf{x} = - \int_D \nabla u \cdot \nabla v + \int_{\partial D} u \nabla v \cdot \hat{n} \, d\mathbf{x}.$$ 

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Remark: just like integration by parts in higher dimensions.
Consider space of smooth functions with domain $D$, satisfying either Dirichlet or Neumann homogeneous boundary conditions. Use inner product

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To compute adjoint of $\Delta$, using Green’s identity twice:

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The integrals on the boundary $\partial D$ vanish because of the boundary conditions. It follows Laplacian is self-adjoint.
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Non-negativity of the eigenvalues

Take inner product of an eigenfunction \( v \) with both sides of the eigenvalue equation \( \mathcal{L}v + \lambda v = 0 \), leading to

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\lambda = -\frac{\langle \mathcal{L}v, v \rangle}{\langle v, v \rangle}, \quad \text{“Rayleigh quotient”}
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Does not determine \( \lambda \), but can be used to estimate it!
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Specialize to our situation (with homogeneous boundary conditions)

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