Fourier’s coffee cup: model as a disk

\[ u_t = D \Delta u, \quad u(a, \theta, t) = u_a, \quad u(r, \theta, 0) = u_0, \]

\( u_a = \) air temperature at boundary, \( u_0 = \) initial coffee temperature
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Now get problem for \( w = u - u_p \) which we can solve:

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Most general solution to this is just superposition of separated solutions

\[ w = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} [A_{nm} \cos(n\theta) + B_{nm} \sin(n\theta)] J_n(\beta_{nm} r / a) e^{-D\beta_{nm}^2 t / a^2} \]
Notice initial condition does not depend on $\theta$, so simplifies to

$$w = \sum_{m=1}^{\infty} A_0 m J_0(\beta_0 m r / a) e^{-D \beta_0^2 m t / a^2}.$$
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Impose initial conditions

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$$\sum_{m=1}^{\infty} A_0m J_0(\beta_0 m r/a) = u_0 - u_a,$$

Recall $J_0(\beta_0 m r/a)$ are orthogonal (with respect to weighted inner product) for different $m$, thus

$$A_0m = \frac{\int_0^a J_0(\beta_0 m r/a)(u_0 - u_a)r \, dr}{\int_0^a J_0^2(\beta_0 m r/a)r \, dr}$$
Still too complicated! Only use term with slowest decay ("ground state approximation")

\[ w \approx A_{01} J_0(\beta_{01} r/a) e^{-D\beta_{01}^2 t/a^2}. \]

It follows that temperature in center is

\[ u(0, t) = u_a + w(0, t) \approx u_a + (u_0 - u_a) e^{-D\beta_{01}^2 t/a^2} \]
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For \( a = 3 \text{cm}, D = .001 \text{cm}^2/\text{sec} \), \( \beta_{01} = 2.404 \), exponential decay rate is \( \exp(-t/t_c) \) where \( t_c = D\beta_{01}^2/a^2 \approx 1000 \text{sec}. \)
Example # 2: Fourier’s Doughnut

Problem: find fundamental (smallest) frequency for wave equation

\[ u_{tt} = c^2 \Delta u \]

on an annulus \( 1 < r < 2 \), subject to boundary conditions

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Recall separated solutions \( u = T(t)v(r, \theta) \) solve \( T'' = -c^2 \lambda T \) and \( \Delta v = -\lambda v \). Since \( T = \cos(c\sqrt{\lambda}t) \) and \( \sin(c\sqrt{\lambda}t) \), frequencies are \( c\sqrt{\lambda} \). We therefore want the smallest eigenvalue.
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Separation \( v = \Theta(\theta)R(r) \) leads to \( \Theta = \cos(n\theta) \) and \( \sin(n\theta) \) as before. For each \( n \), \( R \) solves the Bessel equation

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In this case, we cannot omit the solutions which are singular at the origin, so

\[ R(r) = c_1 J_n(\sqrt{\lambda} r) + c_2 Y_n(\sqrt{\lambda} r) \]
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Eigenvalues are selected by imposing boundary conditions:

\[ 0 = c_1 J_n(\sqrt{\lambda}) + c_2 Y_n(\sqrt{\lambda}), \quad 0 = c_1 J_n(2\sqrt{\lambda}) + c_2 Y_n(2\sqrt{\lambda}). \]
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This linear system has nonzero solutions if determinant is zero:

\[ J_n(\sqrt{\lambda}) Y_n(2\sqrt{\lambda}) = J_n(2\sqrt{\lambda}) Y_n(\sqrt{\lambda}) \]

which is better written as intersection point of graphs

\[ Q_n(\sqrt{\lambda}) = Q_n(2\sqrt{\lambda}), \quad Q_n(x) = \frac{J_n(x)}{Y_n(x)} \]
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Thus smallest eigenvalue is therefore \( \lambda \approx 3.4^2 \).
Consider wave equation with forcing

\[ u_{tt} = c^2 \Delta u + \cos(\omega_0 t), \]

Suppose for some given domain \( \Omega \) and boundary conditions, we already know eigenfunctions \( v_k(x, y) \) and eigenvalues \( \lambda_k \), for \( k = 1, 2, 3, \ldots \). Look for particular solution which has spatial dependence expanded in eigenfunctions
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Look for particular solution which has spatial dependence expanded in eigenfunctions

\[ u_p = \cos(\omega_0 t) \sum_{k=1}^{\infty} A_k v_k(x, y) \]
Resonance in forced oscillations, cont.

Plug into equation (using the fact that $\Delta v_k = -\lambda_k v_k$) to get

$$\sum_{k=1}^{\infty} A_k (\lambda_k c^2 - \omega_0^2) v_k(x, y) = 1.$$
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Just an orthogonal expansion of eigenfunctions, so taking inner products with each eigenfunction gives

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In this case, this means that the particular solution is approximately

$$u_p \approx A_K \cos(\omega_0 t) v_K(x, y).$$  

Resonance “picks out” eigenfunction w/ frequency near $\omega_0$. 
Resonance in forced oscillations, cont.

Resonance for a disk:  Video demonstration

Resonance for a square plate:  Video demonstration