Homework 1B
Math 456/556

Directions: answer using complete sentences. Write each problem on a separate sheet of paper.

Question 1  A. Show that
\[ \langle f, g \rangle = \int_{-1}^{1} x^2 f(x)g(x) \, dx \]
is an inner product on functions \( f : [-1, 1] \rightarrow \mathbb{R} \). (hint: look at the definition!)
B. For which positive integer exponents \( m, n \) will \( x^n, x^m \) be orthogonal with respect to this inner product?

Question 2  A. What is the adjoint of the differential operator (with respect to the standard \( L^2 \) inner product \( \langle f, g \rangle = \int_{0}^{1} f(x)g(x) \, dx \))
\[ \mathcal{L} = x \frac{d^2}{dx^2}, \]
acting on \( C^2_0[0, 1] \) (i.e. twice differentiable functions whose values are zero on the boundaries \( x = 0, 1 \)).
Hint: write down \( \langle \mathcal{L}f(x), g(x) \rangle \) and move the derivatives off \( f(x) \) using integration by parts. You will be left with \( \langle f(x), \mathcal{L}^1 g(x) \rangle \).
B. Instead of the \( L^2 \) inner product, suppose you used the inner product
\[ \langle f, g \rangle = \int_{0}^{1} \frac{f(x)g(x)}{x} \, dx. \]
Show that \( \mathcal{L} \) is self-adjoint with respect to this inner product.

Question 3  A. Solve the Sturm-Liouville eigenvalue problem (i.e. find all eigenvalues and corresponding eigenfunctions)
\[ u''(x) + \lambda u(x) = 0, \]
with boundary conditions \( u(0) = 0, u'(L) = 0 \).
B. One can show that the linear operator \( d^2/dx^2 \) (acting on functions with the correct boundary conditions) is self-adjoint with respect to the usual \( L^2 \) inner product. Show by explicit integration that all the eigenfunctions you found are orthogonal with respect to the same inner product.
C. Since the set of eigenfunctions \( v_n(x) \) is complete, then for any smooth \( f(x) \) with the same conditions, we can write
\[ f(x) = \sum_{n=1}^{\infty} C_n v_n(x). \]
Write down integral formulas for the coefficients \( C_n \). (This is yet another version of a Fourier series.)