Homework 3  
Math 456/556

**Question 1**  A concentration of molecules $u(x, y, t)$ diffuses and reacts, resulting in the following model:

$$u_t = D\Delta u + ru, \quad u(x, 0, t) = u(0, y, t) = u(x, M, t) = u(L, y, t) = 0.$$  

What is the maximum reaction rate $r$ for which $u$ cannot grow exponentially at large times? (hint: look at the separated solution with slowest exponential decay rate)

**Question 2**  Consider the eigenvalue problem $\Delta v + \lambda v = 0$ in the unit square $\{0 < x < 1, 0 < y < 1\}$, with boundary conditions $v = 0$ on the bottom and vertical sides, and $v(x, 1) + v_y(x, 1) = 0$ for the top (this is called the mixed or Robin boundary condition). Find the eigenvalues and corresponding eigenfunctions in terms of solutions of $s + \tan s = 0$. Note this equation cannot be solved analytically, but you should argue graphically that there are infinitely many positive solutions $s_m, m = 1, 2, 3, \ldots$.

**Question 3**  Solve Laplace’s equation for $u(r, \theta, z)$ inside a circular cylinder subject to boundary conditions $u(r, \theta, 0) = 0$ on the bottom, $u(r, \theta, H) = f(r) \sin(7\theta)$ on the top, and $u(a, \theta, z) = 0$ on sides. The function $f(r)$ here is arbitrary; you will ultimately need to find its expansion in terms of certain orthogonal Bessel functions.

**Question 4**  Consider the wave equation

$$u_{tt} = c^2\Delta u$$

on the quarter-circle domain $\{0 < r < a, 0 < \theta < \pi/2\}$, with boundary conditions $u_\theta = 0$ on the straight sides and $u = 0$ on the curved side.

A. Find the characteristic frequencies (i.e. the coefficients of $t$ in the time dependent part). You will need to use the notation $\beta_{nm}$ for the zeros of the Bessel functions.
B. Write down the most general solution resulting from separation of variables.

**Question 5**  Find two solutions of the diffusion equation on the surface of a sphere $u_t = \Delta_s u$, corresponding to two different initial conditions: $u(\phi, \theta, t = 0) = 4 \sin \theta \cos \phi$ and $u(\phi, \theta, t = 0) = 3 \sin^2 \theta \sin(2\phi)$ (Hint: do these look like spherical harmonics? what are the corresponding eigenvalues?).

**Question 6**  Solve the Laplace equation outside the sphere where $r > a$, with boundary condition $u(\phi, \theta, a) = g(\phi, \theta)$. Also assume $u \to 0$ as $r \to \infty$. 