Math 587 Homework 5 Solutions

1. Find a first order expansion for
\[ \epsilon y'' - y^3 = -1 - 7x^2, \quad y(0) = 0, \quad y(1) = 2 \]

Outer Solution: setting \( \epsilon = 0 \) (for leading order approximation) gives \( y_{\text{out}} = (1 + 7x^2)^{1/3} \) where the BC is satisfied at \( x=1 \) and not at \( x=0 \); this means that a boundary layer must be at \( x=0 \).

Inner Solution: using the inner variable \( X = x/\delta \) gives
\[ \epsilon/\delta^2 Y'' - Y^3 = -1 - 7(X\delta)^2 \] where \( \delta = \epsilon^{1/2} \) from dominant balance. The equation then becomes
\[ Y'' - Y^3 + 7X^2 + 1 = 0. \]
Substituting the expansion \( Y = Y_0 + \epsilon^{1/2} Y_1 + \ldots \) yields the leading order equation
\[ Y_0'' - Y_0^3 + 1 = 0. \]

Two ways of finding the boundary layer solution:
(1) Integrate as a Hamiltonian system to give a constant of motion \( C = \frac{1}{2}Y'^2 - \frac{1}{4}Y^4 + Y \). Matching requires \( Y \sim 1 \) as \( X \to \infty \) so \( C = 3/4 \). One can then integrate again (symbolically!!)
(2) Construct a solution geometrically using the 2D phase plane. This amounts to finding an orbit in the first quadrant which starts at \( Y = 0, Y' = \sqrt{(3/4)} \) which connects to the saddle at \( Y = 1, Y' = 0 \).

2. Find the first term expansion for
\[ \epsilon y'' = -(x^2 - 1/4)y', \quad y(0) = 1, \quad y(1) = -1. \]
The equation is in the form \( \epsilon y'' + a(x)y' + b(x)y = 0 \) with \( a(x) = x^2 - 1/4 \) and \( b(x) = 0 \). Since \( a(x) \) has different signs at the boundaries \( a(0) < 0 < a(1) \) and \( a'(x) > 0 \), there must be an interior layer at \( x = 1/2 \) since \( a(1/2) = 0 \).

Outer Solution: setting \( \epsilon = 0 \) (for leading order approximation) gives \( y_{\text{out}}' = 0 \), therefore \( y_{\text{out}} = \text{constant} \). Applying BC’s, \( y_{\text{out}} = 1 \) for \( 0 \leq x < 1/2 \) and \(-1 \) for \( 1/2 < x \leq 1 \).

Inner Solution: using the inner variable \( X = (x - 1/2)/\delta \) gives \( \epsilon/\delta^2 Y'' + X^2 Y' + XY' = 0 \) with \( \delta = \epsilon^{1/2} \) (dominant balance). Substituting the expansion \( Y = Y_0 + \epsilon^{1/2} Y_1 + \ldots \) yields the leading order equation \( Y_0'' + XY_0' = 0 \)
where \( Y_0 = C_1 + C_2 \int_0^X e^{-t^2/2} dt \). Matching inner and outer solutions give 
\( C_1 + \sqrt{\pi/2} C_2 = -1 (x < 1/2, X \to \infty) \) and 
\( C_1 - \sqrt{\pi/2} C_2 = 1 (x > 1/2, X \to -\infty) \). Therefore \( C_2 = -\sqrt{2/\pi}, C_1 = 0 \).

3. Find a two-term expansion
\[ \epsilon y'' = 1 - (y')^2, \quad y(\pm) = 1 \]

Outer Solution: substitute the expansion \( y = y_0 + \epsilon y_1 + \ldots \) and equate powers of \( \epsilon \) to yield \( y_0'^2 = 1 \). It follows that either \( y_0 = |x| \) or \( y_0 = 2 - |x| \) (only the former will match the inner problem). Notice that further corrections will solve \( 0 = -2y_0 y_1' \), so \( y_n = 0 \) for all \( n = 1, 2, 3, \ldots \).

Inner Solution: using the inner variable \( X = x/\epsilon \) gives \( Y'' + 1/\epsilon Y'^2 - \epsilon = 0 \) where the balance is between all three terms. Substituting the expansion \( Y = Y_0 + \epsilon Y_1 + \ldots \) and equating like powers or \( \epsilon \) yields
\[ Y_0'^2 = 0, \quad Y_0'' + 2Y_0'Y_1' = 0, \quad Y_1'' + Y_1'^2 = 1. \]

It follows that \( Y_0 = \text{constant} \) and the order \( \epsilon \) equation yields nothing. The solution to the the third equation is
\[ Y_1 = \ln \left( C_1 + e^{2X} \right) - X + C_2. \]

Matching requires
\[ Y_0 + \epsilon Y_1 \sim y_0 = \epsilon X \]
for \( X \to \pm \infty \) and \( \epsilon \to 0 \). Thus \( Y_0 = 0, C_1 = 1 \) and \( C_2 = 0 \).