Section 7.4, Exercise 18:

Exercise: To find  
\[
\int \frac{x^4 + 12x^3 + 15x^2 + 25x + 11}{x^3 + 12x^2 + 11x} \, dx
\]

**ALGEBRA (elementary).**

\[
\frac{x^4 + 12x^3 + 15x^2 + 25x + 11}{x^3 + 12x^2 + 11x} = \frac{x(x^3 + 12x^2 + 11x) + 4x^2 + 25x + 11}{x^3 + 12x^2 + 11x}
\]

\[
= x + \frac{4x^2 + 25x + 11}{x^3 + 12x^2 + 11x}
\]

(This could also be obtained by long division.)

\[
= x + \frac{4x^2 + 25x + 11}{x(x^2 + 12x + 11)} = x + \frac{4x^2 + 25x + 11}{x(x + 11)(x + 1)}
\]

(Factor denominator.)

**ALGEBRA (partial fractions).**

Now let's focus on doing partial fractions on the rational function in the second term:

\[
\frac{4x^2 + 25x + 11}{x(x + 11)(x + 1)} = \frac{A}{x} + \frac{B}{x + 11} + \frac{C}{x + 1}
\]

Using the quick method discussed in class when we have a linear factor in the denominator, to find \(B\) we multiply by \(x + 11\) and set \(x = -11:\)

\[
B = \frac{4x^2 + 25x + 11}{x(x + 1)} \bigg|_{x = -11} = \frac{4 \cdot 11^2 - 25 \cdot 11 + 11}{(-11)(-11 + 1)} = \frac{20(11)}{-11(-10)} = 2.
\]

Similarly, to find \(A\) we multiply by \(x\) and set \(x = 0:\)

\[
A = \frac{4 \cdot 0^2 + 25 \cdot 0 + 11}{(0 + 11)(0 + 1)} = 1.
\]

And to find \(C\) we multiply by \(x + 1\) and set \(x = -1:\)

\[
C = \frac{4 \cdot (-1)^2 - 25 \cdot (-1) + 11}{(-1)(-1 + 11)} = \frac{-10}{(-10)} = 1.
\]

Combining all this, we have

\[
\frac{x^4 + 12x^3 + 15x^2 + 25x + 11}{x^3 + 12x^2 + 11x} = x + \frac{1}{x} + \frac{2}{x + 11} + \frac{1}{x + 1}
\]

(This can be checked!)

**CALCULUS.** Integrating, we obtain as our final answer:

\[
\int \frac{x^4 + 12x^3 + 15x^2 + 25x + 11}{x^3 + 12x^2 + 11x} \, dx = \frac{1}{2}x^2 + \ln(|x|) + 2\ln(|x+11|) + \ln(|x+1|) + C
\]