1. As defined in the textbook, the general form of a power series is \( \sum_{n=0}^{\infty} C_n (x-a)^n \).

(a) For the polynomial \( 2 + x + x^3 \), determine \( a, C_0, C_1, C_2, C_3, C_4, \) and \( C_5 \).

(b) Is \( 2 + x + x^3 \) a power series?

**SOLUTION.** The polynomial \( 2 + x + x^3 \) is a power series, as are all polynomials. In this case, for the general formula, \( a = 0 \), so we get

\[
\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + \ldots = 2 + x + x^3,
\]

so \( C_0 = 2, C_1 = 1, C_2 = 0, C_3 = 1, C_4 = 0, C_5 = 0 \).

2. For each of the following two sequences, determine whether it converges, and if it does what it converges to. Of course, you should (briefly) explain your answer.

(a) \((-1)^n\)

(b) \(\frac{(-1)^n}{n}\)

**SOLUTION.**

(a) The first sequence does not converge, because it oscillates, \(-1, 1, -1, 1, \ldots\).

(b) The second sequence converges to 0 because the denominator gets arbitrarily large while the numerator is always either 1 or -1.

**COMMENT.** These are not SERIES, they are SEQUENCES. No sums, no alt series test, \ldots.

3. For each of the following two series, determine whether it converges. In each case, justify (explain) your answer by using either the integral test or the alternating series test. In each case, explain how and why the appropriate conditions for the test are satisfied so that you can draw conclusions from the test that you use.

(a) \(\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\)

(b) \(\sum_{n=1}^{\infty} \frac{1}{n}\)

**SOLUTION.**

(a) We can apply the alternating series test to the first series, which is an alternating series. The sequence of terms, \(1/n\), is decreasing and converges to 0. So the series converges.

(b) For the second series, we can use the integral test, since \(f(x) = 1/x\) is a positive, decreasing function:

\[
\int_{1}^{\infty} \frac{1}{x} \, dx = \lim_{b \to \infty} \int_{1}^{b} \frac{1}{x} \, dx = \lim_{b \to \infty} \ln(b) = \infty.
\]

Since the integral diverges, the series diverges.

[OR, use the \(p\)-test with \(p = 1\), which implies divergence since \(p \leq 1\).]

4. Is the series \(\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\) divergent, conditionally convergent, or absolutely convergent? Explain, of course.

**SOLUTION.** The series is conditionally convergent, because it converges but the series of absolute values does not converge. (See problem 3.)
5. Write the following sum in “closed form”, using a known power series or Taylor series.:
\[ \sum_{n=0}^{\infty} \frac{7^n}{n!} \]

**SOLUTION.** \[ \sum_{n=0}^{\infty} \frac{7^n}{n!} = e^7 \] because \[ e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \].

6. Determine the interval of convergence for the following series:
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n^2} (x - 4)^n \]

**SOLUTION.** We have \[ a_n = \frac{(-1)^n}{2^n n^2} (x - 4)^n \], so
\[
\frac{a_{n+1}}{a_n} = \frac{1}{2^{n+1} (n+1)^2} \frac{|x - 4|^{n+1}}{|x - 4|^n} = \frac{1}{2} \frac{n^2}{(n+1)^2} |x - 4|,
\]
which has a limit of \[ \frac{1}{2} |x - 4| \].

The series converges if this limit is < 1 and diverges if this is > 1.

For convergence, \[ \frac{1}{2} |x - 4| > 1 \], so \[ |x - 4| < 2 \].

Thus, the radius of convergence is 2.

At the endpoints of the interval of convergence, i.e., at \( x = 2 \) and \( x = 6 \), the series becomes
\[ \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n^2} (\pm 2)^n = \sum_{n=1}^{\infty} \frac{(\pm 1)^n}{n^2}, \]
which converges by the \( p \)-test with \( p = 2 > 1 \).

So the interval of convergence is the closed interval \([2, 6]\).

7. Using the definition of the Taylor polynomial, obtain the Taylor polynomial of degree three about \( x = 1 \) for the function \( f(x) = 1/x \).

**SOLUTION.** \( f(x) = 1/x, \ f'(x) = -1/x^2, \ f''(x) = 2/x^3, \ f'''(x) = -3 \cdot 2/x^4 \), so
\[
f(1) = 1/1 = 1, \ f'(1) = -1/1^2 = -1, \ f''(1) = 2/1^3 = 2, \ f'''(1) = -3 \cdot 2/1^4 = 3!,
\]
so the third degree Taylor polynomial is
\[
P_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!} (x-1)^n = f(1) + f'(1)(x-1) + \frac{1}{2!} f''(1)(x-1)^2 + \frac{1}{3!} f'''(1)(x-1)^3
\]
\[
= 1 - (x-1) + \frac{1}{2!} 2 (x-1)^2 + \frac{1}{3!} 3! (x-1)^3 = 1 - (x-1) + (x-1)^2 - (x-1)^3
\]