Suppose \((\phi_n)\) is a complete orthogonal sequence of functions on an interval \([a, b]\), so that we can expand any function we’re interested in on \([a, b]\), say \(f\), in the following way:

\[
f = \sum_{n=0}^{\infty} A_n \phi_n. 
\]

More precisely, given such an \(f\), there is a sequence of “constants” \((A_n)\) such that for an appropriate interpretation of convergence of the infinite series, \(f\) is equal to the infinite series.

1. Use the method which we have used several times in class to obtain a formula for the coefficients \(A_n\) in terms of the scalar products \(\langle f, \phi_n \rangle\) and \(\langle \phi_n, \phi_n \rangle\).

   Call this set of equations, equation (1).

Now we want to apply this to the Fourier series on the interval (for convenience) \([-1, 1]\).

In the textbooks’s notation, the Fourier series looks like

\[
f(x) = a_0 \left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} a_n \cos(n \pi x) + \sum_{n=1}^{\infty} b_n \sin(n \pi x). 
\]

Here is one convenient way to make this look like the general form given above:

Define \(A_0 = a_0\), and for every integer \(n \geq 1\), \(A_{2n} = a_n\) and \(A_{2n-1} = b_n\).

Define \(\phi_0(x) = \left\{\frac{1}{2}\right\}\), \(\phi_{2n}(x) = \cos(n \pi x)\), and \(\phi_{2n-1}(x) = \sin(n \pi x)\).

Equivalently, for \(n \geq 0\), \(A_n = a_n/2\) if \(n\) is even, and \(A_n = b_{(n+1)/2}\) if \(n\) is odd.

2. Write similar formulas for \(A_n\) for the two cases where the subscript is even and odd; if you need to treat \(n = 0\) separately, feel free to be wise and do so.

3. Check the correctness of this by separating the sum in equation (0) (as we have often done with power series) into the \(n = 0\) term, and a sum for even \(n\), and a sum for odd \(n\), making the substitutions indicated, and see if you really get the Fourier series as indicated.

4. Assuming everything is OK so far, use your answer for \(A_n\) in Problem 1 (the equation (1)) to obtain the formulas for the coefficients in the Fourier series (2) WITHOUT doing any actual integration. Note the point: You’re supposed to obtain the formula for the coefficients in the Fourier series from the general formula (1), not by doing specific integration for this specific case of orthogonal functions.

(These formulas are given in the book for the general interval \([-l, l]\) and were given in class for the case \(l = 1\). Of course, you should have been able to get the \(l = 1\) case from the general case directly, regardless of what was done in class.)