Suppose \((\varphi_n)\) is a complete orthogonal sequence of functions on an interval \([a, b]\), so that we can expand any function we’re interested in on \([a, b]\), say \(f\), in the following way:

\[
 f = \sum_{n=0}^{\infty} A_n \varphi_n .
\]

More precisely, given such an \(f\), there is a sequence of “constants” \((A_n)\) such that for an appropriate interpretation of convergence of the infinite series, \(f\) is equal to the infinite series.

1. Use the method which we have used several times in class to obtain a formula for the coefficients \(A_n\) in terms of the scalar products \(<f, \varphi_n>\) and \(<\varphi_n, \varphi_n>\).

Call this set of equations, equation (1). (These are the equations for the coefficients \(A_n\).)

Now we want to apply this to the Fourier series on the interval (for convenience) \([-1, 1]\).

In the textbooks’s notation, the Fourier series looks like

\[
 f(x) = a_0 \left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x) .
\]

Here is one convenient way to make this look like the general form given above:

Define \(A_0 = a_0\), and for every integer \(n \geq 1\), \(A_{2n} = a_n\) and \(A_{2n-1} = b_n\).

Define \(\varphi_0(x) = \frac{1}{2}\), \(\varphi_{2n}(x) = \cos(n\pi x)\), and \(\varphi_{2n-1}(x) = \sin(n\pi x)\).

2. There is no Problem 2 (this time).

3. Check the correctness of this by
   - separating the sum in equation (0) at the beginning of this page into the \(n = 0\) term, and a sum for even \(n\), and a sum for odd \(n\) (as we have often done with power series),
   - making the substitutions indicated, and
   - seeing if you really get the Fourier series as indicated.

In other words, you should be able to transform the general equation (0) into the Fourier series (2).

4. Assuming everything is OK so far, use your answers for \(A_n\) in Problem 1 (the equation (1)) to obtain the formulas for the coefficients in the Fourier series (2) WITHOUT doing any actual integration. Note the point: You’re supposed to obtain the formula for the coefficients in the Fourier series from the general formulas (1), not by doing specific integration for this specific case of orthogonal functions.

To finalize this result, you will need to calculate (or to know) the “norms”, \(<\varphi_n, \varphi_n>\), for the specific functions in the Fourir series.

(These formulas are given in the book for the general interval \([-l, l]\) and were given in class for the case \(l = 1\). Of course, you should have be ability to get the \(l = 1\) case from the general case directly, regardless of what was done in class.)