An important part of doing mathematical problems is to recognize connections between different problems (not simply focus on each problem individually), to use appropriate simplifications when appropriate (in particular, NOT when instructed otherwise), and to check answers of calculations or algorithms. Checking is particularly important when (a) the calculations leading to an answer are somewhat complicated, and (b) when there is a simple way to check an answer.

The following problems address these issues as related to the Fourier series problems, Problems 15.1.11 and 15.1.14, as they were extended in the corresponding homework assignment, Lesson 14.

Please approach the following problems from the point of view just described; don’t view them simply as problems for which you are supposed to “turn the crank” and get an answer.

**Regarding Problem 15.1.11.**

The point of the first couple of problems below is the following: It is only necessary to do ONE Fourier series calculation to do all of the series which were requested for this function: The full Fourier series on the original interval \([-\pi, \pi]\), and the cosine, sine, and full Fourier series on the half-interval \([0, \pi]\). And that work itself can be cut in half by thinking about things.

We use \(f\) as the name of the function dealt with in Problem 15.1.11 (this is what it is called in the textbook), which is given on the interval \([-\pi, \pi]\). We will use \(f_0\) to denote the function on \([-\pi, \pi]\) obtained as follows: Restrict the original \(f\) to the “half-interval” \([0, \pi]\) and then construct the ODD extension of this restricted \(f\) to the interval \([-\pi, \pi]\). (This is a slightly different use of the notation \(f_0\) than used in the textbook.) Similarly, \(f_e\) is the function obtained by restricting the original \(f\) to the half-interval and then doing the even extension to the interval \([-\pi, \pi]\).

1. a. Sketch graphs of \(f\) and \(f_0\). Obtain a simple formula, valid on the interval \([-\pi, \pi]\), relating algebraically (the outputs of) the original function \(f\) to the function \(f_0\). (You can either express \(f\) in terms of \(f_0\), or express \(f_0\) in terms of \(f\).) Your formula should look something like the following: \(f_0 = (666f + 75)/23\); this is given just as an idea of what kind of formula we’re talking about.

b. Use this formula to obtain the Fourier sine series on the interval \([0, \pi]\) from the original Fourier series on the interval \([-\pi, \pi]\). (Of course, you are not supposed to do this without explanation, plugging in numbers. Include some words to justify logically what you are doing.)

2. So, for this function, if we have the Fourier series for \(f\) on \([-\pi, \pi]\), we don’t have to do any more integration to obtained the sine series on the half-interval (and vice-versa). The cosine series is even easier: Think about what the function \(f\) looks like and explain how to get the cosine series on \([0, \pi]\) without doing any integration. (You have been given the answer for the cosine series, but explain as if you didn’t have the answer and were trying to get the answer in a simple way. An explanation should include the answer to WHY.)

3. Answer a question similar to Problem 2 just above for the full Fourier series of the restricted function on the half-interval \([0, \pi]\).
4. Variation on Problem 1 above, focusing on only the original function \( f \).

Reminder: “Officially”, to determine the Fourier series, one needs to do (and most students did) two (or three) separate calculations, for the coefficients \( a_n \) of the cosine functions, the coefficients \( b_n \) of the sine functions, and often separately for the coefficient \( a_0 \).

For our given function \( f \), think about the function \( f - \frac{1}{2} \) on the same interval (i.e., \( f(x) - \frac{1}{2} \) for all \( x \) in \([-\pi, \pi]\)); let’s call this function \( f_{\text{shift}} \). (CAUTION: There is a letter \( f \) in the subscript.)

a. What can you say about the evenness or oddness of this new function \( f_{\text{shift}} \)? Given this fact, how could you determine the Fourier series for this new function by calculating only one set of coefficients? (Focus on shifted function and its oddness or evenness.)

b. Perform the calculations suggested in (a) to obtain the Fourier series for \( f_{\text{shift}} \) on the interval \([-\pi, \pi]\), and then use this result to obtain the Fourier series of the original \( f \) on this interval.

The preceding problems focused on how you could simplify your work and implicitly gave you some checks on your answers after you have obtained then (if you didn’t use the simplified techniques to get the answers).

The following problems focus on checking your answers, in the sense of checking some points to see if the series converge the way they should at the given points. Many of the incorrect answers given in homework could have been seen to be wrong by doing such checks.

For some of these checks, you may find it convenient to use the numerical series given to you in an earlier problem (Problem 12 in Section ???).

5. Check the Fourier series for \( f \) at the points \( \pm \pi, \pm \pi/2, \) and \( 0 \) (five points altogether). (Rhetorical question: Would your answer survive this check?) Be sure to explain why the value you get is the value you should get for the point in question, given the nature of the function \( f \).

6. Repeat Problem 5 just above for the sine series on \([0, \pi]\) at the points \( \pi, \pi/2, \) and \( 0 \). (Don’t just repeat what you did before, although the calculations may be very similar; you can use the result of a previous calculation as long as what you are doing is completely clear, but explain why your answer is the correct answer.)

7. Repeat Problem 5 just above for the cosine series on \([0, \pi]\) at the points \( \pi \) and \( 0 \). Your numerical answer may seem obvious, but explain carefully why it is correct for the given function; i.e., suppose you didn’t know what the cosine series was – you should still be able to predict what the values of the cosine series should be at these points.

The comment at the end of Problem 7 can be applied to all three of these problems: You have a function and you have a certain kind of series; you should be able to predict the value the series converges to without knowing the series. Remember, the point is to check the series.