Please read the following instructions. If you agree to abide by, and do abide by, the instructions, sign below and turn in with your exam. If you do not agree to abide by, or do not abide by, the instructions, do not turn in this exam. Exams without a signature will not graded. Any violations of this agreement will be dealt with according to the University’s Code of Academic Integrity.

Instructions for Take Home Exams.

After beginning to read the Take Home Exam (on paper and/or online), I have sought help for this exam, or for material directly related to this exam, from no source, animate or inanimate, other than

- my textbook for Math 425-2, Fall 11,
- notes (including material online) from Math 425-2, Fall 11,

and

- my homework from Math 425-2, Fall 11.

All work submitted is mine and mine alone.

I have read and followed the instructions above.

_____________________________________
Signature

THIS IS AN EXAM. THIS IS AN EXAM. THIS IS AN EXAM.
Previous Homework, for background:
In Exercise 5(a) in Section 6.1, it’s helpful to use the fact that if \( g(x) \leq f(x) \) for all \( x \) in an interval \( J \), then
\[
\inf\{g(x) \mid x \text{ in } J\} \leq \inf\{f(x) \mid x \text{ in } J\}. \tag{1a}
\]
In Exercise 5(b), it’s helpful to use the fact that if \( L(g, P) \leq L(f, P) \) for all partitions \( P \) in a given set \( \Pi \) of partitions of an interval, then
\[
\sup\{L(g, P) \mid P \text{ in } \Pi\} \leq \sup\{L(f, P) \mid P \text{ in } \Pi\}. \tag{2b}
\]
Of course, similar results are true, but irrelevant here, for suprema in the case of the inequality (1a) and for infima in the case of inequality (2b). Let’s refer to these inequalities as (1b) and (2a), respectively.

1. Given real-valued functions \( F \) and \( G \) defined on a set \( S \), these inequalities are consequences of general results for infima [in the case of (1a) and (2a)] and suprema [in the case of (1b) and (2b)] of the sets of values of \( F \) and \( G \) on \( S \). **State and prove rigorously and carefully** the general result for infima, and state (without proof) the general result for suprema.

2. Then, redo Exercise 5 in Section 6.1, making clear and explicit use of the general results just cited. (In particular, when you use the general result, make it clear what the set \( S \) and the functions \( F \) and \( G \) are in the specific cases you are dealing with in Exercise 5.)

**ADDENDUM**

I.a. Read Problem 1 above carefully. **You are given real-valued functions \( F \) and \( G \) defined on a set \( S \).** That is the context in which you should state the theorems asked for in Problem 1 above.

**You are supposed to state and prove a GENERAL result for the infima of the sets of values of \( F \) and \( G \) on \( S \), with appropriate simple additional assumptions on the general \( F \) and \( G \).**

According to the first sentence, this GENERAL result is supposed to have BOTH inequalities (1a) and (2a) as consequences. (Cf. Problem I.b below.)

Similarly, you are supposed to state, without proof, a GENERAL result for suprema which has BOTH inequalities (1b) and (2b) as consequences. This GENERAL result for suprema should be exactly similar to the GENERAL result for infima, with perhaps a slight change in assumptions and with “inf” replaced “sup”.

I.b. Use the GENERAL result for \( \inf \) from Problems 1 and I.a to prove inequality (1a). Make it clear what the set \( S \) and the functions \( F \) and \( G \) are in the specific cases you are dealing with here.

Also, use the GENERAL result for \( \sup \) stated in in Problems 1 and I.a to prove inequality (2b). Make it clear what the set \( S \) and the functions \( F \) and \( G \) are in the specific case you are dealing with here.

When you are done with this, it should be obvious to a reader of your work that you could also prove inequality (1b) and inequality (2a) as simple consequences of the GENERAL theorems in Problems 1 and I.a.

2. After Problem 1b, Problem 2 above is relatively trivial, and you don’t have to redo it, although you should know how.
This problem assumes knowledge of elementary, first semester calculus: The definition of and rules for the derivative, the derivatives of elementary functions, and elementary properties of limits. In particular, we know that the sine function, \( \sin \), and the cosine function, \( \cos \), are bounded.

**I.** Find the limit as \( x \) approaches 0 of \( x^{4/3} \sin(1/x) \). You do not need to give a rigorous proof, but explain.

Define a function \( f \) on the set of real numbers as follows:

For \( x \neq 0 \), \( f(x) = x + x^{4/3} \sin(1/x) \);
For \( x = 0 \), \( f(x) = 0 \).

**II.** Using the techniques of elementary calculus, find the derivative \( f'(x) \) for \( x \neq 0 \).

**III.** Find the limit as \( x \) approaches 0 of \( f'(x) \), if it exists, and explain. (Formal proof not required.)

**IV.** What can you say about the derivative at 0, \( f'(0) \)? Does it exist? Is \( f' \) continuous at 0? Is \( f' \) bounded in a neighborhood of 0? Prove your answers (you can use the results above, or similar results, in your proof.)
Caution: Be careful, and be wise.

**V.** For each natural number \( k \), find the value of \( f'(x) \) at \( x = 1/(2k\pi) \).

**VI.** Comment on the validity of the following, and prove your answer using results above if useful:

If a function \( f \) is differentiable everywhere and, at a point \( a \), \( f'(a) > 0 \), then \( f \) is increasing on a neighborhood \( a \) (i.e., there exists a neighborhood of \( a \) on which \( f \) is increasing).
(We assume that both the domain and the codomain of \( f \) are the set of real numbers.)