Let \( \{a_n\} \) be a sequence of real numbers.

[From Exercise 15 in Sect. 2.1.]
We say that the sequence \( \{a_n\} \) **converges to** \( \infty \) **iff**
For every positive real number \( c \), there exists an index \( N \) such that
for every natural number \( n \geq N \), \( a_n > c \).

[Basic definition.]
We say that the sequence \( \{a_n\} \) **converges to** \( 0 \) **iff**
For every positive real number \( \varepsilon \), there exists a natural number \( N \) such that
for every natural number \( n \geq N \), \( |a_n| < \varepsilon \).

**The ULTIMATE goal of this exercise** is to prove that if a sequence \( \{a_n\} \) of positive numbers has the property that \( \{1/a_n\} \) **converges to** \( 0 \) then the sequence \( \{a_n\} \) converges to \( \infty \).

**Proof.**

0. We are given a sequence \( \{a_n\} \). Suppose the sequence \( \{1/a_n\} \) **converges to** \( 0 \)
   
   *There is nothing for YOU to do here. This is just the beginning of the proof.*

1. Look at the definition of “converges to \( \infty \)” given above. How should/would you start the proof that
   the sequence \( \{a_n\} \) converges to \( \infty \) (using the standard approach to proving universally quantified statements)?

1.a. Are we thinking of the positive number in Step 1 as a large positive number or a small positive number? (I.e., which is the important case in the context of Step 1? This question is about Step 1, not Step 0.)

2.a. Look at the definition of “converges to \( 0 \)” given above. Are we thinking of the positive number \( \varepsilon \) this definition as a large positive number or a small positive number? (I.e., which is the important case in the context of this definition?)

2.b. In order to USE Step 0 and the definition of \( \{1/a_n\} \) “converges to \( 0 \)”, we want to make a clever choice of the universally quantified variable \( \varepsilon \) in that definition. Given the thoughts in Steps 1a. and 2a., how can we get from Step 1 some number to which to apply the definition of
   \( \{1/a_n\} \) “converges to \( 0 \)” ? (I.e., given Step 1, what will we choose for \( \varepsilon \) in order to apply the definition of \( \{1/a_n\} \) “converges to \( 0 \)” ?)

2. Look at the definition of “converges to \( 0 \)” given above. Given the fact that the sequence \( \{1/a_n\} \) **converges to** \( 0 \) and given the positive real number from Step 2.b., what can you conclude from the fact that the sequence \( \{1/a_n\} \) **converges to** \( 0 \) ? (See the last two lines of this definition; don’t give a proof yet, just write down what the definition tells you.)
3. Look at the definition of “converges to ∞” given above. We want to come up with a number \( N \) which satisfies this definition for the given number. From the information given in Step 2, how should we choose the number \( N \) so that the condition given in the definition of “converges to ∞” will be satisfied? (For now, just use some foresight to choose \( N \) appropriately. We will verify that it “works” (third line) in the following steps.)

4. Look at the definition of “converges to ∞” given above. We want to verify that the condition given in the last line of this definition is satisfied for our choice of \( N \) in Step 3. How should/would you start the proof (using the standard approach to proving universally quantified statements) of the last line of the definition?

5. Now, starting as indicated in Step 4, show that the condition given in the last line of the definition of “converges to ∞” is satisfied.

6. What do you conclude (from Steps 1-5) about the sequence given in Step 0?