Let $a$ and $b$ be real numbers with $a < b$.

**Definition of partition**

By a **partition** of the interval $[a, b]$ we mean a function $P$ with the following properties:

- For some natural number $n$, $P : [0, n] \to [a, b]$, where $[0, n] = [0, n] \cap \mathbb{Z} = \{0, 1, \ldots, n\}$
- For every integer $k$ with $0 \leq k \leq n$, let $p_k = P(k)$. Then $a = p_0 < p_1 < \ldots < p_{n-1} < p_n = b$.

Note that the function $P$ is one-to-one (injective). A partition of $[a, b]$ could be alternatively described as a strictly increasing “finite sequence” with $n+1$ terms, with $p_0 = a$ and $p_n = b$.

In keeping with the usual terminology for sequences, we will try to refer to a point $p_k$ as a **term** of the partition $P$, although the word "element" might also be used, even though to speak accurately, $p_k$ is an element of $\text{ran}(P)$, not of $P$.

We denote by $P[a, b]$ the set of all partitions of the interval $[a, b]$.

**Refinements of partitions** (see sketch on reverse or next page)

Given partitions $P$ and $Q$ in $P[a, b]$, we say that $Q$ is a **refinement** of $P$, or that $Q$ is **finer** than $P$, iff

\[
\text{ran}(P) \subset \text{ran}(Q).
\]

Note: Suppose $\text{dom}(P) = \{0, \ldots, m\}$ and $\text{dom}(Q) = \{0, \ldots, n\}$. Given $i$ in $\text{dom}(P)$, we know that $p_i$ is in $\text{ran}(Q)$. Thus we can find $k$ in $\text{dom}(Q)$ such that $p_i = q_k$. Since, as noted above, the partition $Q$ is one-to-one function, there is only one such $k$ in $\text{dom}(Q) = \{0, \ldots, n\}$.

Thus we have defined a function $\rho : \text{dom}(P) = \{0, \ldots, m\} \to \text{dom}(Q) = \{0, \ldots, n\}$ ($\rho$ is the Greek letter rho) such that, in the notation of the preceding paragraph, $k = \rho(i)$; i.e., $p_i = q_\rho(i)$ or $Q(\rho(i)) = P(i)$.

**Partitions induced by refinements**

For the partition $P$ and some appropriate natural number $k$, consider the interval $[p_{k-1}, p_k]$.

Then $[p_{k-1}, p_k] = [\rho(p_{k-1}), \rho(p_k)]$.

By defining $Q_k : \{0, \ldots, \rho(k) - \rho(k-1)\} \to [p_{k-1}, p_k] = [\rho(p_{k-1}), \rho(p_k)]$ by

\[
Q_k(i) = Q(i + \rho(k-1)),
\]

we get a partition $Q_k$ of $[p_{k-1}, p_k] = [\rho(p_{k-1}), \rho(p_k)]$.

**Note:** Be sure to distinguish the function $Q_k$ on $\{0, \ldots, \rho(k) - \rho(k-1)\}$ from the term (value of the function) $q_k = Q(k)$.

Suppose $P$ is a partition of $[a, b]$ and $[c, d]$ is a subinterval of $[a, b]$. We say that a partition $Q$ of $[c, d]$ is **induced by** the partition $P$ iff $\text{ran}(Q) = \text{ran}(P) \cap [c, d]$. We say that the subinterval $[c, d]$ is **determined by** the partition $P$ iff there exists a natural number $k$ in $\text{dom}(P)$ such that $[p_{k-1}, p_k] = [c, d]$. **We have just shown that every partition $Q$ which is finer than $P$ induces a partition, $Q_k$, on any subinterval $[p_{k-1}, p_k]$ which is determined by $P$.**

**Compatible subintervals**

We will say that a subinterval $[c, d]$ of $[a, b]$ is **compatible with** the partition $P$ iff there exist natural numbers $i$ and $k$ in $\text{dom}(P)$ such that $c = p_i$ and $d = p_k$. In other words, $c$ and $d$ are in $\text{ran}(P)$. 