1. (This problem takes place in $\mathbb{R}$, not $\mathbb{R}^n$ for $n > 1$.) Suppose that a power series 
\[ \sum_{k=0}^{\infty} c_k (x - x_0)^k \] converges for a real number $z \neq x_0$. **What can you say about the set $D$,** where 
\[ D = \{ x \mid \sum_{k=0}^{\infty} c_k (x - x_0)^k \text{ converges} \} \]
(E.g.,
- what kind of set is it,
- how “large” is it,
- is there absolute or uniform convergence,
- ...)

**Explain briefly**, based on general known properties of power series. You can use the expression “radius of convergence” if you wish, as long as you use it correctly. You don’t have to prove anything formally, just as long as it is clear that you understand properties of convergence of power series.

**SOLUTION.** There are two possibilities:

1. The series converges for every real number. In this case, the convergence is absolute, and the series converges uniformly on every closed, bounded interval (in fact, on every closed bounded set).

or

2. There exists a positive number $r$ such that 
\[ (x_0 - r, x_0 + r) \subseteq D \subseteq [x_0 - r, x_0 + r] \].

So, in this case, $D$ is a bounded interval centered at $x_0$, and, since the series converges at $z$, $r \geq |z - x_0|$. The series converges absolutely at least at every point in $(x_0 - r, x_0 + r)$, and it converges uniformly on every closed subinterval of $(x_0 - r, x_0 + r)$. (The series may or may not converge at the endpoints of the interval.)
2. For a set $S$,

(a) Explain what is meant by an “interior point” of $S$, using the “open-ball” definition.

(b) Give the definition of the “interior” of $S$, using part (a).

(c) [Note: you will be using the following definition in part (d) below and also in Problem 3, so you might want to look over those problems before giving your final answer here.]

Explain what it means (by definition) to say that $S$ is “open”. You can use the concepts introduced in part (a) or part (b) to give the definition, or give a definition using open balls without using the concept of “interior”.

(d) Prove, using your definitions given above, that the interior of $S$ is an open set. (For convenience, you can assume that int$(S)$ is nonempty, but don’t waste time worrying about this.)

SOLUTION. (a) A point $u$ is an interior point of a set $S$ iff there exists an open ball $B(u, r)$ (of radius $r$ centered at $u$) such that $B(u, r) \subseteq S$.

(b) The interior of a set $S$ is the set of all interior points of $S$ (denoted by int$(S)$).

(c) A set $S$ is open iff every point in $S$ is an interior point of $S$.

(Since interior points are obviously elements of $S$, this is equivalent to int$(S) = S$.)

(d) Proof that the interior of $S$ is an open set.

Suppose $u$ is an interior point of $S$. We want to find an open ball centered at $u$ such that this open ball is a subset of int$(S)$ (not merely a subset of $S$). Since $u$ is an interior point of $S$, we can find an open ball $B(u, r)$ which is a subset of $S$. We will prove that every point in this open ball is in the interior of $S$, which will prove that $u$ is an interior point of $S$. There are two ways to do this:

i. Using previous results: We know that an open ball is an open set. Thus every point $v$ of $B(u, r)$ is contained in an open ball $B(v, \rho)$ centered at $v$ which is a subset of $B(u, r)$. I.e., $B(v, \rho) \subseteq B(u, r) \subseteq S$.

This shows that $v$ is an interior point of $S$, and thus $B(u, r) \subseteq \text{int}(S)$.

Thus, $u$ is an interior point of int$(S)$.

So every point in int$(S)$ is an interior point of int$(S)$, and therefore int$(S)$ is open.

ii. “From scratch”, using only the definitions: Start as above, with $u$, $B(u, r)$, and $v$ in $B(u, r)$. Let $\rho = r - \|v - u\|$. Then $\rho > 0$ since $\|v - u\| < r$. Let $x$ be a point in the ball $B(v, \rho)$. Then, by the triangle inequality,

$\|x - u\| \leq \|x - v\| + \|v - u\| \leq \rho + \|v - u\| = r$

by definition of $\rho$. So $x$ is in $B(u, r)$ and therefore $B(v, \rho) \subseteq B(u, r) \subseteq S$.

This shows that $v$ is an interior point of $S$, and thus $B(u, r) \subseteq \text{int}(S)$.

Thus, $u$ is an interior point of int$(S)$.

So every point in int$(S)$ is an interior point of int$(S)$, and therefore int$(S)$ is open.

NOTE WELL. No point is an “interior point” all by itself. A point may or may not be an “interior point of a set”, but one always needs to be clear what the set if before talking about “interior point”.