Introduction to the mathematical modeling of multi-scale phenomena

Overview

MCB/MATH 303
Below are four **stripe patterns** found in nature. Taken out of context, and in the absence of color, they all look more or less the same. Can you recognize where they come from?

Can you guess the **size** of each pattern?

Can you discuss how many **scales** are visible in each of these patterns?
Patterns are in fact very common in nature and in the laboratory.

Examples include
- Hexagonal and traveling wave patterns due to bioconvection;
- Spiral and target structures found in slime moulds, in chemical reactions, or in the heart;
- Hexagonal geological structures such as columnar joints.

Why do so many patterns look alike?

The answer lies in a universal theory of pattern formation.
Below are four stripe patterns found in nature. Taken out of context, and in the absence of color, they all look more or less the same. Can you recognize where they come from?

Can you guess the size of each pattern?

Can you discuss how many scales are visible in each of these patterns?
Other examples of multi-scale systems

- **Hydrodynamic turbulence** observed in fluids is a typical example of a system that involves a variety of scales.
  - In **three dimensions**, energy cascades down to smaller scales until dissipation kicks in.
  - In **two dimensions**, such as in for instance soap films, an inverse cascade of energy is observed, leading to the formation of large-scale vortices.

- The following examples involve effects at the **micro- and nano-scales**.
  - The Lycurgus cup
  - The edelweiss
  - The gecko
This module will also discuss the following topics:

- **Self-similarity** and fractals.
- **Dimensional analysis** and scalings.
- Application to **self-similar solutions** of partial differential equations.
- **Diffusion** at the microscopic and macroscopic levels.
- Generalization of the above to **bacterial motion**.
- Time permitting, extension of these concepts to ideas of **anomalous diffusion** and foraging behaviors.
Assignments

- Due **Thursday, August 28**: Quiz on plagiarism and self-plagiarism (D2L).

- Due **Tuesday, September 2**: Written essay (D2L).

- **Today in class**: identify one of the topics discussed above, to be presented in class on Tuesday, September 16.

- When you are done, start learning **how to use MATLAB**.
Two different types of “scale-free” systems

- **Pattern-forming systems**
  1. Pattern-forming systems display a *variety of periodic structure* (such as stripes, hexagons, squares, etc), which occur over a wide range of scales.
  2. The universal theory of pattern formation is somewhat *scale-free*, in the sense that it applies to phenomena that occur at different scales.
  3. It is however built on the fact that *most patterns have at least two scales*, one which is “fast” (the period of the pattern), and one which is “slow” (the scale at which the pattern changes in space, or time).

- **Self-similar systems**
  We now turn to another type of “scale free” systems, those which look identical at all scales. They are called *self-similar systems*. 
Self-similarity and fractals

We will start our exploration of multi-scale systems with a discussion of self-similarity.

- **Self-similar** objects look identical at each scale (or magnification).
- **Fractals** are self-similar, as is illustrated in the *Julia Sets* MATLAB applet.
- We will spend some class time exploring and understanding *Julia sets*. 
Diffusion-limited aggregation (DLA) is a model for the formation of aggregates proposed by Witten and Sander in 1981.

In two dimensions, this process gives rise to branched clusters of fractal dimension $d = 1.71$.

The process may be extended to three dimensions, where it gives rise to very complex clusters.

I am sure you all noticed that the three-dimensional DLA cluster looks like a tree.

In fact, fractals are used to create artificial landscapes which look quite realistic.

The picture on the right was created with the software Teragen.
I am sure you all noticed that the three-dimensional DLA cluster looks like a tree.

In fact, fractals are used to create artificial landscapes which look quite realistic.

The picture on the right was created with the software Teragen.
Objects found in nature, like the *Romanesco broccoli* pictured on the right, present self-similar structures.

Self-similarity is also observed in laboratory experiments, such as the tearing of a plastic sheet.

A simplified version of this experiment can be done in class.

A direct analogy can be made between torn plastic sheets and some types of leaves or petals.
Objects found in nature, like the *Romanesco broccoli* pictured on the right, present self-similar structures.

Self-similarity is also observed in laboratory experiments, such as the tearing of a plastic sheet.

A simplified version of this experiment can be done in class.

A direct analogy can be made between torn plastic sheets and some types of leaves or petals.
These photographs, from The British Museum's web site, show the Lycurgus cup, made of dichroic glass. The cup changes color (from green to red) when it is held up to the light. This is due to colloidal particles of silver and gold imbedded into the glass.
The edelweiss

The velvety aspect of the edelweiss is due to the presence of filaments (of about 10 µm in diameter) that protect the flower from damage due to UV light. These filaments are themselves covered with nano-scale fibers (of average diameter 180 nm).

The toes of a gecko are covered with micro-hairs which are themselves covered with possibly hundreds of nano-hairs. The structure of these hairs allows the gecko to climb on walls by means of dry adhesion.

The fabrication of synthetic gecko hair may lead to the commercialization of new types of residue-less adhesives. It is also used in the development of wall climbing robots (see for instance the Carnegie Mellon NanoRobotics Lab web site).
Patterns in nature
Patterns in nature
Patterns in nature
Bioconvection

Layer of "heavy" bacteria

Fluid with nutrients


