Name:

☐ Check box if you are part of a study group

Notes:
1. Show all of your work, present it neatly, and explain what you are doing. In particular, write complete sentences, either using words or mathematical symbols.
2. You will only receive credit for the work that is shown.
3. It is more important to leave out one or two sub-questions and do all of the others in depth, than to do a little bit of each and finish none.
4. Two pages of formulas for Fourier series and transforms and for Laplace transforms are attached to this exam.
5. Question 9 is a bonus question and is worth 10 points.

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1) [12 points] Find all of the values of $z$ for which $z^6 = -64$. Show all your work. Give an exact, but simplified, answer for each root.
2) Consider the function \( u(x, y) = \sin(x) \cosh(y) \), where \( x \) and \( y \) are real variables.

a) [4 points] Show that the function \( u(x, y) \) is harmonic. Show all your work.

b) [8 points] Find a function \( v(x, y) \) such that \( u \) and \( v \) are harmonic conjugates. Explain what you are doing.
c) [3 points] Write $u(x,y) + i v(x,y)$ as a function of $z = x + i y$. Show all your work.
3) [20 points] Solve the following system of differential equations

\[
\begin{align*}
\frac{dx_1}{dt} &= 3x_1 - x_2 \\
\frac{dx_2}{dt} &= 8x_1 - 3x_2
\end{align*}
\]

with initial conditions \(x_1(0) = 2, \ x_2(0) = 3\).

You may use the method of your choice, but you have to show all of your work. If you use Laplace transforms, remember that \(\mathcal{L}(e^{at})(s) = \frac{1}{s-a}\).
4) [20 points] Find the Fourier transform of \( f(x) = x e^{-\alpha |x|} \), where \( \alpha > 0 \). Show all your work.
5) [10 points] Consider the following polynomials (they are two Legendre polynomials):

\[ P_2(x) = \frac{1}{2} (3x^2 - 1), \quad P_3(x) = \frac{1}{2} (5x^3 - 3x). \]

Calculate the dot product of \( P_2 \) and \( P_3 \), and show that these two polynomials are orthogonal. Here, the dot product of two functions \( u \) and \( v \) is defined as

\[ \langle u, v \rangle = \int_{-1}^{1} u(x) v(x) \, dx. \]
6) [35 points] Solve the heat equation,
\[
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 \leq x \leq 2, \quad t \geq 0,
\]
with boundary conditions \(u(0,t) = 0\) and \(u(2,t) = 0\), and initial condition \(u(x,0) = \sin(\pi x) + 7 \sin(2\pi x) - 8 \sin(4\pi x)\). Clearly identify all of the steps in solving this equation. Show all your work.
7) Consider the following vectors in $\mathbb{R}^3$.  

\[
U_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}.
\]

a) [3 points] Show that $U_1$, $U_2$ and $U_3$ are orthogonal to one another.
b) [10 points] Show that $U_1, U_2$ and $U_3$ form a basis of $\mathbb{R}^3$. Explain your reasoning.
c) [10 points] Expand the vector $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ on the orthogonal basis $\{U_1, U_2, U_3\}$. In other words, write $X$ as a linear combination of $U_1$, $U_2$ and $U_3$. Show all your work.
8) Consider the function $f(x) = e^x$, defined on the interval $-1 \leq x \leq 1$.

a) [17 points] Find the complex Fourier series of the function $f$. Show all your work.

b) [8 points] What is the Fourier series of $f$ equal to at $x = 1$? Explain.
9) **Bonus questions (10 points):** Calculate the Laplace transform of \( f(t) = t \)
and use this information to find the inverse Laplace transform of \( \frac{1}{s^4} \).

Generalize your result and find the inverse Laplace transform of \( \frac{1}{s^n} \), for 
\( n = 2, 3, 4, \ldots \).