Math 322                   Midterm I

Name: ____________________________________________

Notes:
1. Show all of your work, present it neatly, and explain what you are doing. In particular, write complete sentences, either using words or mathematical symbols.
2. You will only receive credit for the work that is shown.
3. You can score a maximum of 130 points on this test. So there are 10 “bonus” points.
4. It is more important to leave out a problem and do all of the others in depth, than to do a little bit of each problem and finish none.

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1. Consider the function \( f(z) = \frac{i}{z-1} \).

   a) [5 points] Write down the Cauchy-Riemann equations for an analytic function \( u(x,y) + i \, v(x,y) \).

   b) [12 points] Use the Cauchy-Riemann equations to show that the function \( f \) above is analytic everywhere, except at \( z = 1 \).

   c) [3 points] Is the function \( f \) entire? Why or why not?
2. Consider the function \( u(x,y) = x (x^2 - 3 y^2) \).
   a) [5 points] Show that \( u \) is harmonic.
   
   b) [12 points] Find the harmonic conjugate of \( u \), and call this function \( v(x,y) \). Explain what you are doing.
   
   c) [3 points] Express \( u(x,y) + i v(x,y) \) as a function of the variable \( z = x + i y \). Show all of your work.
3. Evaluate the following expressions and write the result in the form $a + i b$, where $a$ and $b$ are real. Give \textbf{exact answers} (do not approximate them numerically). If a function is multi-valued, give \textbf{all of the possible values} of that function.

a) [3 points] $\ln(\sqrt{3} + i)$

b) [3 points] $\ln(\sqrt{3} + i)$

c) [3 points] $\cosh(2 + i\pi)$

d) [3 points] $\exp(7i\pi + 1)$

e) [8 points] $\sqrt[3]{-2 + 2i}$
4. Consider the function \( g(z) = \frac{1}{z} \).

a) [4 points] Given that it involves a term in \( z \), do you expect \( g(z) \) to be differentiable at \( z = 1 \)? Why or why not?

b) [4 points] Express the following difference quotient in terms of \( \Delta x \) and \( \Delta y \):

\[
\Delta y: Q(\Delta x, \Delta y) = \frac{g(1 + \Delta z) - g(1)}{\Delta z}, \quad \text{where } \Delta z = \Delta x + i \Delta y.
\]

Do not try to separate the real part of \( Q \) from its imaginary part.
c) [10 points] Calculate the limit of $Q(\Delta x, \Delta y)$ as $\Delta z \to 0$, along the line of equation $\Delta y = t \Delta x$. Show all your work.

d) [4 points] Using the result of part c), show that the function $g(z)$ is not differentiable at $z = 1$. Justify your answer.

e) [3 points] Is the function $g$ analytic at $z = 1$? Why or why not?

6. [10 points] Use De Moivre’s formula to express \( \cos(4x) \) in terms of \( \cos(x) \) and \( \sin(x) \).
7. [10 points] Is the following set of vectors linearly independent? Why or why not?

\[
\begin{bmatrix}
2 & 0 & 1 \\
4 & 2 & 1 \\
7 & 5 & 1
\end{bmatrix}
\]
8. [10 points] Find a basis of the column space of the matrix \( A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 2 & 1 \\ 7 & 5 & 1 \end{bmatrix} \).

Note that the columns of \( A \) are the same vectors as in Problem 7 above. Explain what you are doing.
9. Consider the vector \( U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \).

   a) [5 points] Find two vectors, \( U_1 \) and \( U_2 \), whose entries are **all non-zero**, and which, together with the vector \( U_3 \) above, form a basis for \( \mathbb{R}^3 \).

   b) [5 points] Explain why the set \( \{U_1, U_2, U_3\} \) that you have found is a basis of \( \mathbb{R}^3 \).