Name:

Notes:
1. Show all of your work, present it neatly, and explain what you are doing. In particular, write complete sentences, either using words or mathematical symbols.
2. You will only receive credit for the work that is shown.
3. You can score a maximum of 130 points on this test. So there are 10 “bonus” points.
4. It is more important to leave out a problem and do all of the others in depth, than to do a little bit of each problem and finish none.

<table>
<thead>
<tr>
<th>Question</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score</td>
<td>/20</td>
<td>/20</td>
<td>/20</td>
<td>/25</td>
<td>/5</td>
<td>/10</td>
<td>/10</td>
<td>/10</td>
<td>/10</td>
<td>/120</td>
</tr>
</tbody>
</table>
1. Consider the function \( f(z) = \frac{1}{z-1} \).

a) [5 points] Write down the Cauchy-Riemann equations for an analytic function \( u(x,y) + i\,v(x,y) \).

\[
\begin{align*}
ux & = vy, \\
vx & = -uy
\end{align*}
\]

are the Cauchy-Riemann equations for \( u(x,y) + i\,v(x,y) = f(z) \).

b) [12 points] Use the Cauchy-Riemann equations to show that the function \( f \) above is analytic everywhere, except at \( z = 1 \).

\[
\begin{align*}
ux(x,y) + iv(x,y) = f(z) = \frac{i}{z-1} = \frac{i}{(x-1)+iy} &= \frac{i((x-1)-iy)}{(x-1)^2+y^2} \\
 &= \frac{y}{(x-1)^2+y^2} + i\frac{x-1}{(x-1)^2+y^2}
\end{align*}
\]

So \( u(x,y) = \frac{y}{(x-1)^2+y^2} \) and \( v(x,y) = \frac{x-1}{(x-1)^2+y^2} \).

Now, \( ux = \frac{-2(x-1)y}{((x-1)^2+y^2)^2} \) and \( vy = \frac{-2(x-1)y}{((x-1)^2+y^2)^2} \)

So \( ux = vy \).

Moreover, \( uy = \frac{(x-1)^2+y^2-2y^2}{[(x-1)^2+y^2]^2} = \frac{(x-1)^2}{[(x-1)^2+y^1]^2} \)

and \( vx = \frac{(x-1)^2+y^1-2(x-1)^2}{[(x-1)^2+y^1]^2} = \frac{-(x-1)^2+y^2}{[(x-1)^2+y^1]^2} = -uy \)

So \( f(z) \) satisfies the Cauchy-Riemann equations everywhere, except at \( z = 1 \). It is analytic, except at \( z = 1 \).

c) [3 points] Is the function \( f \) entire? Why or why not?

The function is not entire since it is not analytic at \( z = 1 \).
2. Consider the function \( u(x, y) = x(x^2 - 3y^2) \).
   
   a) [5 points] Show that \( u \) is harmonic.
   
   \[
   u_x = 3x^2 - 3y^2 \quad u_{xx} = 6x \quad \text{so} \quad u_{xx} + u_{yy} = 6x - 6x = 0
   
   u_y = -6xy \quad u_{yy} = -6x
   
   \text{Since } \Delta u = 0, \text{ the function } u \text{ is harmonic.}
   
   b) [12 points] Find the harmonic conjugate of \( u \), and call this function \( v(x, y) \). Explain what you are doing.
   
   \( u(x, y) = x^3 - 3xy^2 \). We look for a function \( v \) such that \( u + iv \) is analytic. With the Cauchy-Riemann equations, we have \( u_x = v_y \) and \( u_y = -v_x \).
   
   \[
   u_x = v_y \Rightarrow 3x^2 - 3y^2 = v_y
   
   \Rightarrow v(x, y) = 3x^2y - y^3 + h(x)
   
   \Rightarrow v_x = 6xy + h'(x)
   
   \text{Since we need } v_x = -u_y, \text{ we have } 6xy + h'(x) = 6xy
   
   \text{So } h'(x) = 0 \text{ i.e. } h = c \text{ constant.}
   
   \text{We choose } c = 0 \text{ and get } \boxed{v(x, y) = 3x^2y - y^3}.
   
   c) [3 points] Express \( u(x, y) + iv(x, y) \) as a function of the variable \( z = x + iy \). Show all of your work.
   
   \[
   u + iv = x^3 - 3xy^2 + i(3x^2y - y^3) = (x + iy)^3 = z^3.
   
   3
3. Evaluate the following expressions and write the result in the form \( a + i \, b \), where \( a \) and \( b \) are real. Give **exact answers** (do not approximate them numerically). If a function is multi-valued, give **all of the possible values** of that function.

a) [3 points] \( \ln(\sqrt{3} + i) = \ln |\sqrt{3} + i| + i \, \text{arctan} \left( \frac{1}{\sqrt{2}} \right) \)
\[ = \ln (2) + i \cdot \frac{\pi}{4} . \]

b) [3 points] \( \ln(\sqrt{3} + i) = \ln (2) + i \cdot \frac{\pi}{6} + 2\pi i \cdot \pi \quad \rho \in \mathbb{Z} . \)

c) [3 points] \( \cosh(2 + i\pi) = \frac{e^{2 + i\pi} + e^{-2 - i\pi}}{2} = \frac{e^{2} (-1) + e^{-2} (-1)}{2} \)
\[ = - \cosh (2) . \]

d) [3 points] \( \exp(7i\pi + 1) = e^{[\cos (7 \pi) + i \cdot \sin (7 \pi)]} \)
\[ = e^{(-1 + 0)} = -e . \]

e) [8 points] \( \sqrt[3]{-2 + 2i} = \sqrt[3]{2} e^{-i \pi/4} \quad \text{or} \quad \sqrt[3]{2} e^{\frac{-i \pi}{4} + i \frac{2n \pi}{3}} \quad \text{or} \quad \sqrt[3]{2} e^{\frac{i \pi}{4} + i \frac{2n \pi}{3}} \)

Indeed, we need to solve \(-2 + 2i = z^{3} \), since \(-2 + 2i = \sqrt{8} e^{-i \pi/4} \), this reads \( z^{3} = \sqrt{8} e^{-i \pi/4} \quad \text{i.e.} \quad z = \sqrt[3]{\sqrt{8} e^{-i \pi/4} + \rho \frac{2\pi}{3}} \quad \rho = 0, 1, 2 . \)
4. Consider the function \( g(z) = \frac{1}{\bar{z}} \).

   a) [4 points] Given that it involves a term in \( \bar{z} \), do you expect \( g(z) \) to be differentiable at \( z = 1 \)? Why or why not?

   We do not expect \( g(z) \) to be differentiable since \( \bar{z} \) is not analytic in \( z \).

   b) [4 points] Express the following difference quotient in terms of \( \Delta x \), and

   \( \Delta y: Q(\Delta x, \Delta y) = \frac{g(1+\Delta z) - g(1)}{\Delta z} \), where \( \Delta z = \Delta x + i \Delta y \). Do not try to separate the real part of \( Q \) from its imaginary part.

   \[
   Q(\Delta x, \Delta y) = \frac{g(1+\Delta z) - g(1)}{\Delta z} = \frac{1}{\Delta z} \left[ \frac{1}{1+\Delta z} - 1 \right]
   \]

   \[
   = \frac{1}{\Delta x + i \Delta y} \left[ \frac{1}{1+\Delta x - i \Delta y} - 1 \right] = \frac{1}{\Delta x + i \Delta y} \cdot \frac{-\Delta x + i \Delta y}{1+\Delta x - i \Delta y}
   \]

   \[
   = \frac{-\Delta x + i \Delta y}{(\Delta x + i \Delta y)(1+\Delta x - i \Delta y)}
   \]
c) [10 points] Calculate the limit of $Q(\Delta x, \Delta y)$ as $\Delta z \to 0$, along the line of equation $\Delta y = t \Delta x$. Show all your work.

With $\Delta y = t \Delta x$, we have

$$Q(\Delta x, t \Delta x) = \frac{-\Delta x + i t \Delta x}{(\Delta x + i t \Delta x)(1+\Delta x - i t \Delta x)} = \frac{-1 + i t}{(1 + iT)(1 + \Delta x - i t \Delta x)}$$

Since $\Delta z = \Delta x + i \Delta y = \Delta x + i t \Delta x = (1 + iT) \Delta x$,
the limit $\Delta z \to 0$ along the line $\Delta y = t \Delta x$ corresponds to the limit $\Delta x \to 0$ in the expression of $Q(\Delta x, t \Delta x)$.

As $\Delta x \to 0$, $Q(\Delta x, t \Delta x) \to \frac{-1 + i t}{1 + iT} = \frac{(-1 + iT)(1 - iT)}{1 + t^2}$

$$= \frac{-1 + 2i t + t^2}{1 + t^2}$$

i.e.

$$\lim_{\Delta x \to 0} Q(\Delta x, t \Delta x) = \frac{-1 + 2i t + t^2}{1 + t^2}.$$

d) [4 points] Using the result of part c), show that the function $g(z)$ is not differentiable at $z = 1$. Justify your answer.

Since the above limit depends on $t$, i.e. depends on the path followed to approach $z = 1$, the quantity $Q(\Delta x, \Delta y)$ does not have a limit as $z \to 1$. Therefore, $g$ is not differentiable at $z = 1$.

e) [3 points] Is the function $g$ analytic at $z = 1$? Why or why not?

Since $g$ is not differentiable at $z = 1$, it is not analytic at $z = 1$. 

6

For instance, \( \arg(z) \) is multi-valued. Given a particular \( z \), we can write \( \arg(z) = \text{Arg}(z) + 2\pi n \), where \( n \in \mathbb{Z} \) is arbitrary.

6. [10 points] Use De Moivre’s formula to express \( \cos(4\theta) \) in terms of \( \cos(\theta) \) and \( \sin(\theta) \).

De Moivre’s formula is \( e^{i\theta} = (e^{i\theta})^n \), i.e.

\[
\cos(n\theta) + i\sin(n\theta) = [\cos(\theta) + i\sin(\theta)]^n.
\]

With \( n = 4 \), this becomes

\[
\cos(4\theta) + i\sin(4\theta) = [\cos(\theta) + i\sin(\theta)]^4
\]

\[
= [\cos^2(\theta) + 2i\sin(\theta)\cos(\theta) - \sin^2(\theta)]^2
\]

\[
= \cos^4(\theta) - 4\sin^2(\theta)\cos^2(\theta) + \sin^4(\theta) + 4i\sin(\theta)\cos(\theta)
\]

\[
- 2\cos^2(\theta)\sin^2(\theta) - 4i\sin^3(\theta)\cos(\theta)
\]

\[
= [\cos^4(\theta) - 6\sin^2(\theta)\cos^2(\theta) + \sin^4(\theta)] + 4i[\sin(\theta)\cos^3(\theta) - \sin^3(\theta)\cos(\theta)] = \cos(4\theta) + i\sin(4\theta)
\]

By taking the real part of each side of the above equation, we see that

\[
\cos(4\theta) = \cos^4(\theta) + \sin^4(\theta) - 6\cos^2(\theta)\sin^2(\theta).
\]
7. [10 points] Is the following set of vectors linearly independent? Why or why not? \(
\begin{bmatrix}
2 & 0 & 1 \\
4 & 2 & 1 \\
7 & 5 & 1
\end{bmatrix}
\)

\[
\begin{bmatrix}
2 \\
4 \\
7
\end{bmatrix} - 2 \begin{bmatrix}
1 \\
1 \\
2
\end{bmatrix} - \begin{bmatrix}
0 \\
2 \\
5
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}, \quad \text{the 3 vectors are not linearly independent.}
\]
8. [10 points] Find a basis of the column space of the matrix \( A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 2 & 1 \\ 7 & 5 & 1 \end{bmatrix} \).

Note that the columns of \( A \) are the same vectors as in Problem 7 above. Explain what you are doing.

\[
\text{Span } \left\{ \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \left\{ C_1 \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} + C_2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, C_1, C_2, C_3 \in \mathbb{R} \right\}
\]

Since \( \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \), any linear combination of the 3 vectors is in fact a linear combination of \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \). In other words,

\[
\text{Span } \left\{ \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} = \left\{ C_1 \left( 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right) + C_2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \right. \\
C_1, C_2, C_3 \in \mathbb{R} \left. \right\}
\]

\[
= \left\{ (2C_1 + C_3) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (C_1 + C_2) \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, C_1, C_2, C_3 \in \mathbb{R} \right\}
\]

\[
= \left\{ A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + B \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, A, B \in \mathbb{R} \right\} = \text{span } \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}.
\]

Since \( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \) are not proportional, they are linearly independent, and therefore form a basis of the column space of \( A \).
9. Consider the vector \( U_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \).

a) [5 points] Find two vectors, \( U_1 \) and \( U_2 \), whose entries are all non-zero, and which, together with the vector \( U_3 \) above, form a basis for \( \mathbb{R}^3 \).

\[ U_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad U_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \]

are such that \( \{ U_1, U_2, U_3 \} \) is a linearly independent set.

Indeed, \( c_1 U_1 + c_2 U_2 + c_3 U_3 = 0 \) \( \iff \) \[
\begin{align*}
&c_1 + 2c_2 = 0 \\
&2c_1 - c_2 = 0 \\
&c_1 + c_2 + c_3 = 0
\end{align*}
\]

\( \iff \) \[
\begin{align*}
&c_1 + 4c_1 = 0 \\
&c_2 = 2c_1 \\
&3c_1 + c_3 = 0
\end{align*}
\]

\( \iff \) \[
\begin{align*}
&c_1 = 0 \\
&c_2 = 0 \\
&c_3 = 0
\end{align*}
\]

b) [5 points] Explain why the set \( \{ U_1, U_2, U_3 \} \) that you have found is a basis of \( \mathbb{R}^3 \).

Since \( \mathbb{R}^3 \) is 3-dimensional, any basis of \( \mathbb{R}^3 \) has exactly 3 vectors. Since \( \{ U_1, U_2, U_3 \} \) is a linearly independent set, the 3 vectors \( U_1 \), \( U_2 \) and \( U_3 \) must span \( \mathbb{R}^3 \). If they did not, we would be able to find a basis of \( \mathbb{R}^3 \) with more than 3 vectors, which contradicts the fact that every basis has exactly 3 vectors.

Since \( U_1 \), \( U_2 \) and \( U_3 \) are linearly independent and span \( \mathbb{R}^3 \), they form a basis of \( \mathbb{R}^3 \).