Probabilistic combinatorics and lattice gas estimates

William G. Faris

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Abstract

The talk gives an application of lattice gas theory to probabilistic combinatorics, following a recent paper of Scott and Sokal. There is a large collection of bad events; the problem is to show that there is some non-zero probability that none of them occur. The result is that this probability is bounded below by the partition function of the dependency graph of the events. A subsequent talk will give Dobrushin’s explicit lower bound for this partition function.

1 The inclusion-exclusion principle

The inclusion-exclusion principle says that if

\[ f(S) = \sum_{T : S \subseteq T} g(T). \]  

then

\[ g(T) = \sum_{S : T \subseteq S} (-1)^{|S| - |T|} f(S). \]  

Here is a proof. Fix \( T \) and compute

\[
\sum_{S : T \subseteq S} (-1)^{|S| - |T|} f(S) = \sum_{S : T \subseteq S' : S \subseteq T'} (-1)^{|S| - |T'|} g(T')
\]

\[ = \sum_{T' : T \subseteq T'} g(T') \sum_{S : T \subseteq S' \subseteq T'} (-1)^{|S| - |T'|}. \]  

However

\[
\sum_{S : T \subseteq S' \subseteq T'} (-1)^{|S| - |T'|} = \prod_{x \in T' \setminus T} (1 - 1) = \delta_{T, T'}. \]  

So

\[
\sum_{S : T \subseteq S} (-1)^{|S| - |T|} f(S) = \sum_{T' : T \subseteq T'} g(T') \delta_{T, T'} = g(T). \]
2 The standard lattice gas model

In the following the set $X$ is a finite non-empty set. Let $P_2(X)$ be the two-element subsets of $X$. A graph $G$ with vertex set $X$ is a subset of $P_2(X)$. For $x \in X$, let $\Gamma(x)$ be the vertices adjacent to $x$, and let $\Gamma^*(x) = \Gamma(x) \cup \{x\}$.

The usual lattice gas model takes $X$ as a set of sites. At each site there is at most one particle. Two particles may not occupy sites that are adjacent in the graph $G$. Thus subset $S \subseteq X$ is compatible if $P_2(S) \cap G = \emptyset$. The partition function is a sum over compatible subsets of $X$:

$$Z(w) = \sum_{Y: P_2(Y) \cap G = \emptyset} \prod_{y \in Y} w_y. \quad (6)$$

This is a function of the activities $w_x$ for $x$ in $X$. Consider a subset $\Lambda \subseteq X$. We write $Z_\Lambda(w) = Z(1_{\Lambda} w)$ for the partition function where the activities are set equal to zero outside $\Lambda$, so in effect one is only summing over compatible subsets of $\Lambda$.

The standard lattice gas probability model is the following. If $S$ is a compatible subset, then the probability that the set $O$ of occupied sites is such that there are particles in $S$ and no particles in $X \setminus S$ is

$$P[O = S] = \frac{\prod_{x \in S} w_x}{Z(w)}. \quad (7)$$

If $S$ is not compatible, then its probability is zero. It follows that the probability that for $T \subseteq \Lambda$ compatible there are particles in $T$ but not in $\Lambda \setminus T$ is

$$P[O \cap \Lambda = T] = \frac{\left(\prod_{x \in T} w_x\right) Z_{X \setminus \Lambda \setminus \Gamma^*(T)}(w)}{Z(w)}. \quad (8)$$

In particular, if $\Lambda = X$ we recover

$$P[O = T] = \frac{\prod_{x \in T} w_x}{Z(w)}. \quad (9)$$

If $\Lambda = T$ we get

$$P[T \subseteq O] = \frac{\left(\prod_{x \in T} w_x\right) Z_{X \setminus \Gamma^*(T)}}{Z(w)}. \quad (10)$$

Another useful quantity is when $T = \emptyset$, then

$$P[O \cap \Lambda = \emptyset] = \frac{Z_{X \setminus \Lambda}(w)}{Z(w)}. \quad (11)$$

A special case when $\Lambda = X$ and $T = \emptyset$ is

$$P[O = \emptyset] = \frac{1}{Z(w)}. \quad (12)$$
The fundamental equation satisfied by the partition function is

$$Z_\Lambda(w) = Z_{\Lambda \setminus x}(w) + w_x Z_{\Lambda \setminus \Gamma^*}(w). \quad (13)$$

This equation may also be written as

$$\frac{Z_\Lambda(w)}{Z(w)} = \frac{Z_{\Lambda \setminus x}(w)}{Z(w)} + \frac{w_x Z_{\Lambda \setminus \Gamma^*}(w)}{Z(w)}. \quad (14)$$

In the standard particle interpretation this equation says that the sum of the probability of no particle at \(x\) or in the complement of \(\Lambda\) plus the probability of a particle at \(x\) and no particle in the complement of \(\Lambda\) is the probability of no particle in the complement of \(\Lambda\). However the following arguments use a different probability model associated with this partition function.

3 The new lattice gas model

In the new model the activities \(w_x = -p_x\) are negative. However they are supposed to be small enough so that the appropriate partition functions are still positive.

The goal is to have that for \(S\) compatible the probability \(f(S)\) there are particles at every point in \(S\) (and possibly elsewhere) is

$$f(S) = \hat{P}[S \subseteq O] = \prod_{x \in S} p_x. \quad (15)$$

For \(S\) incompatible \(f(S) = 0\).

From this one can determine the probability \(g_\Lambda(T)\) that there are particles on \(T \subset \Lambda\) with no particles on \(\Lambda \setminus T\) (and no specification on \(X \setminus \Lambda\)). In fact, for \(S \subset \Lambda\) we have

$$f(S) = \sum_{T \subseteq S \subseteq \Lambda} g_\Lambda(T). \quad (16)$$

Hence

$$g_\Lambda(T) = \sum_{S \subseteq T \subseteq \Lambda} (-1)^{|S| - |T|} f(S) = (-1)^{|T|} \sum_{T \subseteq S \subseteq \Lambda} \prod_{x \in S} (-p_x), \quad (17)$$

where the sum is over \(S\) that are compatible. After a little calculation, this reduces to

$$g_\Lambda(T) = \hat{P}[O \cap \Lambda = T] = \left(\prod_{x \in T} p_x\right) Z_{\Lambda \setminus \Gamma^*}(\Lambda)(-p), \quad (18)$$

for \(T\) compatible, and zero for \(T\) not compatible. If this is positive, then we have a probability model.

In particular, if \(\Lambda = X\) we get

$$g(T) = g_X(T) = \hat{P}[O = T] = \left(\prod_{x \in T} p_x\right) Z_{X \setminus \Gamma^*}(T)(-p) \quad (19).$$
for $T$ compatible, zero otherwise. If $\Lambda = T$ is compatible we recover
\begin{equation}
\hat{P}[T \subseteq O] = \prod_{x \in T} p_x. \tag{20}
\end{equation}

Another useful quantity is when $T = \emptyset$, then
\begin{equation}
Q_\Lambda = g_\Lambda(\emptyset) = \hat{P}[O \cap \Lambda = \emptyset] = Z_\Lambda(-p). \tag{21}
\end{equation}

A special case when $\Lambda = X$ and $T = \emptyset$ is
\begin{equation}
Q_X = g(\emptyset) = \hat{P}[O = \emptyset] = Z(-p). \tag{22}
\end{equation}

The fundamental equation satisfied by the partition function is
\begin{equation}
Z_{\Lambda \setminus x}(-p) = Z_\Lambda(-p) + p_x Z_{\Lambda \setminus \Gamma^*(x)}(-p). \tag{23}
\end{equation}

In the new particle interpretation this equation says that the sum of the probability of no particle in $\Lambda$ plus the probability of a particle at $x$ but with no other particles in $\Lambda$ is the probability of no particle in $\Lambda \setminus x$.

4 Comparison of the two models

It seems strange that the mathematics of partition functions can lead to two such different probability models. However the relation becomes clear when one looks at the independent case. Then we are just looking at independent success-failure trials indexed by the set $X$. In this case, the partition function is
\begin{equation}
Z(w) = \sum_Y \prod_{y \in Y} w_y = \prod_x (1 + w_x). \tag{24}
\end{equation}

In the standard statistical mechanical interpretation
\begin{equation}
P[O = S] = \frac{\prod_{x \in S} w_x}{Z(w)} = \prod_{y \in S} \frac{w_y}{1 + w_y} \prod_{y \in X \setminus S} \frac{1}{1 + w_y}. \tag{25}
\end{equation}

We see that in this interpretation the activity $w_x$ has the interpretation of the odds ratio, the ratio of the probability $w_x/(1 + w_x)$ of success to the probability $1/(1 + w_x)$ of failure.

In the interpretation in the paper of Scott and Sokal the probability is
\begin{equation}
\hat{P}[O = T] = \left( \prod_{x \in T} p_x \right) Z_{X \setminus T}(-p) = \prod_{x \in T} p_x \prod_{x \in X \setminus T} (1 - p_x). \tag{26}
\end{equation}

So in this interpretation the activity $-p_x$ has the interpretation that $p_x$ is the actual probability of success. Then $1 - p_x$ is the probability of failure. This shows that the minus sign is quite natural. It also shows why there should be a restriction on the magnitude of the $p_x$, so that $1 - p_x$ never becomes negative.
5 Lower bound on the probability of a bad event

There is a large collection of bad events $A_x$ for $x \in X$, each with small probability. The problem is to show that there is some non-zero probability that none of them occur.

Example: Say that $P[A_x] \leq p_x$. (27)

The most primitive lower bound is

$$P[\bigcap_{x \in X} A_x^c] = 1 - P[\bigcup_{x \in X} A_x] \geq 1 - \sum_{x \in X} p_x.$$ (28)

Unfortunately, this quickly gets negative.

Example: Suppose that the $A_x$ are independent with $P[A_x] \leq p_x < 1$. In this case,

$$P[\bigcap_{x \in X} A_x^c] \geq \prod_{x \in X} (1 - p_x) > 0.$$ (29)

This is the kind of lower bound we would like to have, only without the supposition of independence.

Consider numbers $p_x \geq 0$. Consider the partition function condition that $Z(w) > 0$ whenever $-p_x \leq w_x \leq 0$ for all $x$.

**Theorem 1** Let $X$ be a finite non-empty set. For each $x$ in $X$, let $A_x$ be an event in the probability space. Let $P_2(X)$ be the collection of all 2 element subsets of $X$. Let $G \subset P_2(X)$ be a graph with vertex set $X$. Suppose that the $p_x$ satisfy the partition function condition. Suppose that for each $x$ and each $Y \subseteq X \setminus \Gamma^*(x)$ we have

$$P[A_x | \bigcap_{y \in Y} A_y^c] \leq p_x.$$ (30)

Then

$$P[\bigcap_{x \in X} A_x^c] \geq Z(-p) > 0.$$ (31)

For the proof it is helpful to think of this as a particle system of an unusual kind. Let $B_x$ be the event that there is a particle at $x$. We take $\hat{P}[B_x] = p_x$.

A subset $S \subset X$ is said to be compatible if every pair of distinct points in $S$ is not in the graph. For every compatible $S \subset X$ we take

$$f(S) = \hat{P}[\bigcap_{x \in S} B_x] = \prod_{x \in S} p_x.$$ (32)

Thus $f(S)$ is the probability that there is a particle at each point of $S$ in the new particle system described above.

Now write $Q_\Lambda = \hat{P}[\bigcap_{x \in \Lambda} B_x^c] = Z_\Lambda(-p)$ be the probability in the new particle system of no particles in $\Lambda$. Let $P_\Lambda = P[\bigcap_{x \in \Lambda} A_x^c]$ be the probability that we wish to bound. Then it is sufficient to prove that $P_\Lambda/Q_\Lambda$ is increasing in $\Lambda$. 


Now $Q$ satisfies the fundamental equation
\[ Q_{\Lambda \cup \{y\}}(-p) = Q_{\Lambda}(-p) - p_x Q_{\Lambda \setminus \Gamma(y)}(-p). \] (33)

On the other hand, by hypothesis
\[ P_{\Lambda \cup \{y\}} = P_{\Lambda} - P[A_y \cap \bigcap_{x \in \Lambda} A_x]. \] (34)
\[ \geq P_{\Lambda} - P[A_y \cap \bigcap_{x \in \Lambda \setminus \Gamma(y)} A_x]. \] (35)
\[ \geq P_{\Lambda} - p_y P_{\Lambda \setminus \Gamma(y)}. \] (36)

It then follows that
\[ P_{\Lambda \cup \{y\}} Q_{\Lambda} - Q_{\Lambda \cup \{y\}} P_{\Lambda} \geq p_y [P_{\Lambda} Q_{\Lambda \setminus \Gamma(y)} - Q_{\Lambda} P_{\Lambda \setminus \Gamma(y)}] \geq 0. \] (37)

6 Lower bound on the partition function

Let $X$ be a set. Let $P_2(X)$ be the set of 2 element subsets of $X$. Let $G \subseteq P_2(X)$ be a graph. The corresponding partition function is a sum over subsets of $X$ given by
\[ Z(w) = \sum_{Y : P_2(Y) \cap G = \emptyset} \prod_{y \in Y} w_y. \] (38)

It is a polynomial in the activities $w_x$ for $x$ in $X$.

**Theorem 2 (Dobrushin)** Consider $w_x$ with $|w_x| \leq p_x$ for each $x$. For each $x$ in $X$ let $\Gamma(x)$ be the neighbors of $x$ in the graph $G$. Suppose there are numbers $r_x$ with $0 \leq r_x < 1$ and
\[ p_x \leq r_x \prod_{y \in \Gamma(x)} (1 - r_y). \] (39)

Then
\[ Z(w) \geq \prod_{x \in X} (1 - r_x) > 0. \] (40)

**Theorem 3 (Lovász lemma)** For each $x$ there is a number $r_x$ with $0 \leq r_x < 1$. The hypothesis is that for each $Y \subseteq X \setminus \Gamma^*(x)$ we have
\[ P[A_x \mid \bigcap_{y \in Y} A_y^c] \leq r_x \prod_{y \in \Gamma(x)} (1 - r_y). \] (41)

The Lovász lemma concludes that
\[ P[\bigcap_{x \in X} A_x^c] \geq \prod_{x \in X} (1 - r_x). \] (42)

The idea is that for each fixed $x$, $r_x$ is an index of collective propensity for badness. In the hypothesis this index should not be too close to zero, but it should also not be too close to one.
The Kotecký-Preiss condition

The Kotecký-Preiss condition is that the activities satisfy $|w_x| \leq p_x$, where for some $A_x \geq 0$ we have the estimate

$$\sum_{y \in \Gamma^*(x)} p_y e^{A_y} \leq A_x. \quad (43)$$

Dobrushin writes this in the form

$$\exp\left(\sum_{y \in \Gamma(x)} p_y B_y + p_x B_x \right) \leq B_x. \quad (44)$$

He shows that this implies the somewhat weaker Dobrushin condition

$$1 - p_x \exp\left(\sum_{y \in \Gamma(x)} p_y B_y \right) \geq \exp(-p_x B_x). \quad (45)$$

Set $r_x = 1 - \exp(-p_x B_x)$. This is equivalent to

$$p_x \leq r_x \prod_{y \in \Gamma(x)} (1 - r_y). \quad (46)$$