(1a) 10 points Using integration by parts, show every step of your calculation to find the antiderivative $\int z^2 \ln(z) \, dz$

Hint: the answer is $\frac{1}{4} z^3 \ln(z) - \frac{1}{3} z^3 + c.$

\[ u = \ln(z) \quad v' = z^2 \]
\[ u' = \frac{1}{z} \quad v = \frac{1}{3} z^3 \]
\[ = \frac{1}{3} z^3 \ln(z) - \int \left( \frac{1}{z} \right) \left( \frac{1}{3} z^3 \right) \, dz \]
\[ = \frac{1}{3} z^3 \ln(z) - \frac{1}{3} \int z^2 \, dz \]
\[ = \frac{1}{3} z^3 \ln(z) - \frac{1}{3} \cdot \frac{3}{9} + c \]

(1b) 10 points If we have a function $f(x)$ for which we know that $\int_1^8 f(x) \, dx = 24$, find the value of $\int_0^{\ln(2)} e^{3t} f(e^{3t}) \, dt$.

Hint: Try a substitution.

\[ W = e^{3t} \]
\[ dW = 3e^{3t} \, dt \]
\[ W(t=0) = e^0 = 1 \]
\[ W(t=\ln(2)) = e^{3\ln(2)} = e^{\ln(2^3)} = e^{\ln(8)} = 8 \]

\[ = \frac{1}{3} \int_1^8 f(W) \, dW = \frac{24}{3} = 8 \]
(2) 10 points Evaluate the integral. Be sure to show enough work so that I can see what you did!

\[ \int \frac{v}{\sqrt{1 + v^2}} \, dv = \frac{1}{2} \int \frac{1}{\sqrt{w}} \, dw \]

\[ = \frac{1}{2} \left( \frac{w^{1/2}}{1/2} \right) + C \]

\[ = w^{1/2} + C \]

\[ = \sqrt{1 + y^2} + C \]

(3) 15 points A scientist drops his cellphone into a deep hole. The following table describes the velocity \( V(t) \) at various times.

<table>
<thead>
<tr>
<th>( t ) (seconds)</th>
<th>0</th>
<th>250</th>
<th>500</th>
<th>750</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V(t) ) (m/sec)</td>
<td>0</td>
<td>-5</td>
<td>-10</td>
<td>-20</td>
<td>-50</td>
</tr>
</tbody>
</table>

The distance traveled by the phone in 1000 seconds is \( \int_0^{1000} V(t) \, dt \). Use the trapezoidal rule to estimate the distance travelled.

\[ \text{LEFT} \cdot -250 \left( 0 + 5 + 10 + 20 \right) \]

\[ \text{RIGHT} \cdot -250 \left( 5 + 10 + 20 + 50 \right) \]

\[ \text{Trap} \cdot -250 \left( 0 + 10 + 20 + 40 + 50 \right) \]

\[ = \frac{-250}{2} \left( 120 \right) = -250 \times 60 \]

\[ = -15,000 \]

Phone traveled 15,000 meters down the hole.
(4) 10 points Use the trigonometric substitution \( y = 2 \sin(\theta) \) to rewrite the integral. You do not need to evaluate the integral, but you should simplify the integrand.

\[
\int \frac{1}{y^2 \sqrt{4 - y^2}} \, dy = \int \frac{1}{4 \sin^2 \theta \sqrt{4 - 4 \sin^2 \theta}} \cdot 2 \cos \theta \, d\theta
\]

\[
= \int \frac{2 \cos \theta}{4 \sin^2 \theta \cdot 2 \cdot \sqrt{1 - \sin^2 \theta}} \, d\theta = \int \frac{2 \cos \theta}{8 \sin^2 \theta \cdot \cos \theta} \, d\theta
\]

\[
= \frac{1}{4} \int \frac{1}{\sin^2 \theta} \, d\theta
\]

(5) 15 points Use partial fractions to evaluate the following integral.

\[
\int \frac{1}{t^2 - 5t + 6} \, dt.
\]

\[
\frac{1}{t^2 - 5t + 6} = \frac{1}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2}
\]

some algebra shows that \( A = 1 \) and \( B = -1 \).

\[
\int \frac{1}{t^2 - 5t + 6} \, dt = \int \frac{1}{t-3} - \frac{1}{t-2} \, dt = \ln|t-3| + \ln|t-2| + C
\]
(6) 15 points Given the following formula from the Integral Tables,
\[ \int \frac{1}{\sqrt{x^2 \pm a^2}} \, dx = \ln |x + \sqrt{x^2 \pm a^2}| + C. \]
evaluate the following integral:
\[ \int \frac{1}{\sqrt{t^2 - 2t + 5}} \, dt. \]

Complete the square
\[ \int \frac{1}{\sqrt{(t-1)^2 + 4}} \, dt, \]
\[ x = t - 1, \]
\[ dx = dt, \]
\[ \int \frac{1}{\sqrt{x^2 + 4}} \, dx = \ln |x + \sqrt{x^2 + 4}| + C, \]
\[ = \ln |t - 1 + \sqrt{(t-1)^2 + 4}| + C. \]

(7) 7 points Does the following improper integral converge or diverge? You do not need to evaluate the integral, but you do need to justify your answer with intuition or an inequality.
\[ \int_1^\infty \frac{(x-1)^2}{x^4 + x^2} \, dx. \]

The guiding behavior:
\[ \frac{x^2}{x^4 + x^2} = \frac{1}{x^2}, \]
\[ \int_1^\infty \frac{1}{x^2} \, dx \text{ is convergent.} \]

So the intuition is that \( \int_1^\infty \frac{(x-1)^2}{x^4 + x^2} \, dx \) is convergent.

An inequality:
\[ (x-1)^2 \leq x^2 \text{ if } x > 1 \]
\[ \frac{1}{x^4 + x^2} \leq \frac{1}{x^4}. \]
\[ \frac{(x-1)^2}{x^4 + x^2} \leq \frac{x^2}{x^4}. \]
\[ \frac{x^2}{x^4} = \frac{1}{x^2}. \]

Therefore the integral in question is convergent.
Evaluate the following definite integral. Note: if the integral diverges, explain whether it diverges to $+\infty$ or $-\infty$.

$$\int_{1}^{5} (x-1)^{-3/2} \, dx$$

\[\lim_{a \to 1^+} \int_{a}^{5} (x-1)^{-3/2} \, dx = \lim_{a \to 1^+} \left[ \frac{(x-1)^{-1/2}}{-1/2} \right]_{a}^{5} \]

\[= \lim_{a \to 1^+} \left( -2(x-1)^{-1/2} \right) \bigg|_{a}^{5} = \lim_{a \to 1^+} \left( -2(4)^{-1/2} + 2(a-1)^{-1/2} \right) \]

\[= \lim_{a \to 1^+} \left( \frac{-2}{\sqrt{2}} + 2(a-1)^{-1/2} \right) \]

\[= \lim_{a \to 1^+} \left( -1 + \frac{2}{\sqrt{a-1}} \right) = +\infty \]

So the integral diverges to $+\infty$. 
