Problem set 1, Math 413-513 Spring 2003

This problem set is devoted to examples in which a change of basis in a vector space is the key to the simple solution of an important problem. Let \( P_n \) denote the vector space of polynomials of degree less than or equal to \( n \) with real coefficients. As mentioned in class this is a vector space with dimension \( n + 1 \) and \( \{1, x, x^2, \ldots, x^n\} \) is a basis. The simplest interpolation problem for polynomials is to specify \( n + 1 \) points \((x_j, y_j)\) for \( j = 1, \ldots, n \) with all the \( x_j \) distinct and to seek a polynomial \( p \) of degree \( n \) which passes through all these points,

\[
p(x_j) = y_j \quad \text{for} \quad j = 1, \ldots, n + 1.
\]

It is plausible that these \( n + 1 \) equations determine the \( n + 1 \) coefficients of \( p \) but the simplest way to see this is to introduce a different basis for \( P_n \), the Lagrange interpolation basis described in your text. The \( j^{th} \) element in this basis is,

\[
\ell_j(x) = \prod_{k \neq j} \frac{x - x_k}{x_j - x_k}.
\]

The property which makes this basis useful in the interpolation problem is that the \( \ell_j(x) \) are all clearly polynomials of degree \( n \) with \( \ell_j(x_k) = 0 \) if \( k \neq j \) and \( \ell_j(x_j) = 1 \). The solution of the interpolation problem (1) is then,

\[
p(x) = \sum_{j=1}^{n+1} y_j \ell_j(x).
\]

It is worth recalling the argument that shows that the \( \ell_j \) are linearly independent (and hence a basis!). If the linear combination of the \( \ell_j \) given by (2) is zero as a function then in particular \( p(x_j) = 0 \) for \( j = 1, \ldots, n + 1 \).

But then \( y_j = p(x_j) = 0 \) and it follows that the \( \ell_j \) are linearly independent (this argument is almost magically simple). There is another way to look at the Lagrange basis which is instructive. Let \( E_j \) be the linear functional defined on \( P_n \) by,

\[
E_j(p) = p(x_j).
\]

\( E_j \) just evaluates the polynomial \( p \) at the point \( x_j \). Then

\[
E_j(\ell_k) = \delta_{jk} = \begin{cases} 
1 & \text{for } j = k \\
0 & \text{for } j \neq k
\end{cases},
\]

and \( E_j \) and \( \ell_j \) are dual bases (\( E_j \) gives the coordinates for \( \ell_j \)).

1. Determine the Lagrange interpolation basis of \( P_2 \) for the three points \( x_1 = -h \), \( x_2 = 0 \) and \( x_3 = h \) (three evenly spaced points with spacing \( h \)). Use this basis to answer the following questions.

(a) Suppose that you have three data points \((x_1, y_1), (x_2, y_2), \) and \((x_3, y_3)\) which represent a discrete “approximation” to a differentiable function (such approximations are important in computer modeling of differential equations). Find an approximation to the value of the second derivative of this function at the center point \( x_2 = 0 \) by first calculating the quadratic polynomial \( p(x) \) which passes through these three points and then evaluating \( p''(0) \) (the second derivative of \( p \) at \( 0 \)). Do you get the same result if the three data points are altered by changing the \( x \) coordinates to \( x_1 = a - h, x_2 = a, \) and \( x_3 = a + h \) without changing the \( y \) values? The expression you obtain in this way is called a finite difference approximation to \( \frac{d^2}{dx^2} \).

(b) Suppose that you want to approximate the area under the “curve” which has the discrete representation \((x_1, y_1), (x_2, y_2)\) and \((x_3, y_3)\). Find the quadratic polynomial \( p(x) \) which passes through these three points (you did this in part (a)) and evaluate,

\[
\int_{-h}^{h} p(x) \, dx.
\]

Show that the result doesn’t change if \( x_1 = a - h, x_2 = a \) and \( x_3 = a + h \) with the \( y \) values unchanged. Apply this result to the more general problem of estimating \( \int_{a}^{b} f(x) \, dx \). Divide the interval \([a, b]\) into \( 2n \) equal segments of size \( h = \frac{b-a}{2n} \) and consider the discrete approximation to \( f(x) \) obtained by evaluating \( f(x) \) at the points \( x_j = a + jh \) for \( j = 0, 1, \ldots, 2n \). Find an approximation to \( \int_{a}^{b} f(x) \, dx \) which depends only on
the $2n + 1$ values $f(x_j)$ by starting at $a$ and adding together the approximations obtained above for the intervals $(a, a + 2h), (a + 2h, a + 4h)$ and etc. The result is called Simpson’s rule which is a popular numerical integration scheme.

2. Consider the following polynomials in $P_3$.

$$
\begin{align*}
    s_1(x) &= (2x + 1)(x - 1)^2 = 2x^3 - 3x^2 + 1, \\
    s_2(x) &= x(x - 1)^2 = x^3 - 2x^2 + x, \\
    s_3(x) &= -(2x - 3)x^2 = -2x^3 + 3x^2, \\
    s_4(x) &= x^2(x - 1) = x^3 - x^2.
\end{align*}
$$

Show that $\{s_j\}$ is a basis for $P_4$ by observing that the linear functionals,

$$
\begin{align*}
    F_1(p) &= p(0), \\
    F_2(p) &= p'(0), \\
    F_3(p) &= p(1), \\
    F_4(p) &= p'(1),
\end{align*}
$$

are coordinate functions for $\{s_j\}$. That is $F_j(s_k) = \delta_{jk}$. Now argue as in the case of interpolation polynomials that if,

$$
p(x) = \sum_{k=1}^{4} a_j s_j(x) = 0,
$$

then $a_1 = p(0) = 0$, $a_2 = p'(0) = 0$ and etc. Being able to specify derivatives allows one to draw curves through discrete points which don’t have any sharp turns (with continuous derivatives). Use the basis $\{s_j\}$ to solve the following problem. Find a cubic function $p_1(x)$ defined on the interval $[0,1]$ and a cubic function $p_2(x)$ defined on the interval $[1,2]$ so that the function $f(x)$ you get by putting these two functions together to give a function on the interval $[0,2]$ satisfies the following conditions,

(a) The graph of $f$ passes through the three points $(0,2)$, $(1,3)$, and $(2,1)$.

The slope of $f$ at 0 is equal to the slope of the line joining the first two points $(0,2)$ and $(1,3)$. The slope of $f$ at 1 is equal to the slope of the line joining $(0,2)$ with $(2,-1)$ and the slope of $f$ at 2 is equal to the slope of the line joining $(1,3)$ with $(2,-1)$.

Hint: To find the appropriate polynomial $p_2(x)$ on the second interval work with the functions $s_j(x - 1)$ rather than $s_j(x)$.

The functions you are working with here are called cubic splines and are used in computer graphics programs to draw smooth curves passing through discrete points.