Solutions to Homework 6, Math 413-513.

1. (a) Substituting \( u_\lambda(x) = \alpha \cos \lambda x + \beta \sin \lambda x \) into the boundary conditions \( u_\lambda(0) = u_\lambda(L) = 0 \) one finds \( \alpha = 0 \) and \( \beta \sin \lambda L = 0 \). Both \( \alpha \) and \( \beta \) can’t be \( 0 \) or the solution \( u_\lambda(x) \) is not interesting. Thus \( \beta \neq 0 \) and we must have \( \sin \lambda L = 0 \). The solutions to this equation are \( \lambda L = 0, \pm n\pi \) for \( n \) an integer. Only the positive integers give distinct non zero solutions \( u_\lambda(x) = \sin(n\pi x/L) \).

(b) The boundary conditions lead to \( \beta = 0 \) and \( \lambda \sin \lambda L = 0 \). In this case the root \( \lambda = 0 \) gives the non zero eigenfunction \( u_0(x) = 1 \). The other roots \( \lambda L = n\pi \) lead to the eigenfunctions \( u_\lambda(x) = \cos(n\pi x/L) \).

(c) The boundary conditions translate into the linear system of equations for \( \alpha \) and \( \beta \),

\[
\begin{bmatrix}
\cos \lambda L - \delta \sin \lambda L & \sin \lambda L + \delta \cos \lambda L \\
\lambda & 0
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

For this equation to have a non-zero solution \( [\alpha, \beta]^T \) the determinant of the matrix on the left must vanish. A little algebra shows that this equation is,

\[
\tan \lambda L = \frac{\lambda(\gamma - \delta)}{1 + \gamma \delta \lambda^2}.
\]

Plotting the two sides independently one can see that this equation will typically have an infinite number of solutions for \( \lambda \) that get closer and closer to the roots of \( \tan(\lambda L) = 0 \) as \( \lambda \to \infty \).

2. (a) One must compute,

\[
T^n \begin{bmatrix}
a \\
b
\end{bmatrix} := \begin{bmatrix}
-2 & -1 \\
1 & 0
\end{bmatrix}^n \begin{bmatrix}
a \\
b
\end{bmatrix}.
\]

The characteristic equation is \( (\lambda + 1)^2 = 0 \). Thus the generalized eigenspace associated with \( \lambda = -1 \) is two dimensional and hence equal to all of \( \mathbb{C}^2 \). Every vector is thus a root vector even \( [a, b] \). Thus,

\[
T^n \begin{bmatrix}
a \\
b
\end{bmatrix} = (-1 + (T + I))^n \begin{bmatrix}
a \\
b
\end{bmatrix} = (-1)^n \begin{bmatrix}
a \\
b
\end{bmatrix} + n(-1)^{n-1}(T + I) \begin{bmatrix}
a \\
b
\end{bmatrix}.
\]

We don’t need to use any more terms in the expansion since \( (T + I)^2 = 0 \). Thus \( f_n = (-1)^n b + n(-1)^{n-1}(a + b) \).

(b) The characteristic polynomial for,

\[
T = \begin{bmatrix}
3 & -3 & 1 \\
1 & -1 & 0 \\
0 & 1 & -1
\end{bmatrix},
\]

is \( -(\lambda - 1)^3 \). Thus the root space for \( \lambda = 1 \) is 3 dimensional. Hence every vector in \( \mathbb{C}^3 \) is a root vector. Since \( (T - I)^3 = 0 \) we see that,

\[
T^n \begin{bmatrix}
a \\
b \\
c
\end{bmatrix} = (I + (T - I))^n \begin{bmatrix}
a \\
b \\
c
\end{bmatrix} = \begin{bmatrix}
a \\
b \\
c
\end{bmatrix} + n(T - I) \begin{bmatrix}
a \\
b \\
c
\end{bmatrix} + \frac{n(n-1)}{2}(T - I)^2 \begin{bmatrix}
a \\
b \\
c
\end{bmatrix}.
\]

A short calculation shows that

\[
f_n = c + n(b - c) + \frac{n(n-1)}{2}(a - 2b + c).
\]

This will typically grow like \( n^2 \) instead of \( \lambda^n \) unless \( a - 2b + c = 0 \).
3. The critical value \( R_c \) occurs at the point at which the characteristic polynomial experiences a transition between a pair of real roots and a complex conjugate pair of roots. This transition takes place when the discriminant of the quadratic is 0 which you can check takes place at

\[
R_c = 2\sqrt{\frac{L}{C}}
\]

The solutions at \( R = 0, 2 \) and 3 are,

\[
\begin{bmatrix}
\cos t \\
\sin t
\end{bmatrix},
\begin{bmatrix}
e^{-t} \\
e^{-t} 
\end{bmatrix},
\begin{bmatrix}
\frac{\lambda_{+}}{\sqrt{5}} \\
\frac{1}{\sqrt{5}}
\end{bmatrix} - \begin{bmatrix}
\frac{\lambda_{-}}{\sqrt{5}} \\
\frac{1}{\sqrt{5}}
\end{bmatrix},
\]

where,

\[
\lambda_{\pm} = \frac{-3 \pm \sqrt{5}}{2}.
\]

4. The eigenvalues and eigenvectors for the relevant matrix are, \(-2i, [1, 0, -1]^T\) and \((-2 \pm \sqrt{2})i, [1, -1, 1]^T\).

You might have noticed that these eigenvectors are orthogonal to one another – this is not accidental. In any case the initial vector can be expressed as a linear combination of eigenvectors just so,

\[
\begin{bmatrix}
0 \\
1
\end{bmatrix} = \frac{\sqrt{2}}{4}\begin{bmatrix}
1 \\
\sqrt{2}
\end{bmatrix} - \frac{\sqrt{2}}{4}\begin{bmatrix}
1 \\
-\sqrt{2}
\end{bmatrix},
\]

which leads to the solution of the differential equation,

\[
\psi(t) = \frac{\sqrt{2}}{4}e^{(-2+\sqrt{2})it} \begin{bmatrix}
1 \\
\sqrt{2}
\end{bmatrix} - \frac{\sqrt{2}}{4}e^{-(2+\sqrt{2})it} \begin{bmatrix}
1 \\
-\sqrt{2}
\end{bmatrix}.
\]

From this one easily computes,

\[
\psi_1(t) = \frac{e^{-2it}}{\sqrt{2}} \sin \sqrt{2}t,
\]

\[
\psi_2(t) = \frac{e^{-2it}}{\sqrt{2}} \cos \sqrt{2}t,
\]

\[
\psi_3(t) = \frac{e^{-2it}}{\sqrt{2}} \sin \sqrt{2}t.
\]

Thus \( |\psi_1(t)|^2 = |\psi_3(t)|^2 = \frac{1}{2} \sin^2 \sqrt{2}t \), and \( |\psi_2(t)|^2 = \cos^2 \sqrt{2}t \). The plot of these two functions shows that the particle starts off with probability 1 at the central site. After a period \( t = \frac{\pi}{2\sqrt{2}} \) the probability of being at the central site decreases to 0 while the probability of being at either of the two end sites increases to 1/2. In the next period of length \( \frac{\pi}{2\sqrt{2}} \) the particle washes up back to the central site with probability 1 and the whole thing starts all over again.