Comments on Homework 2

1.3.24 The map \( x \to \frac{2}{\sqrt{1 - |x|^2}} \) maps the open unit disk onto \( \mathbb{R}^n \). Compose this with the inverse of stereographic projection from the north pole to get,

\[
\phi(x) = \left(2\sqrt{1 - |x|^2}x, 2|x|^2 - 1\right) \in \mathbb{R}^{n+1}.
\]

This is a continuous map from the closed disk, \( D^n \), onto the sphere \( S^n \). Every point on the boundary of \( D^n \) is mapped to the north pole \((0,1)\). Thus \( \phi \) induces a continuous map from \( D^n / S^{n-1} \) onto \( S^n \). It is obvious by construction (and easy to check directly) that the induced map is one to one. Since \( D^n / S^{n-1} \) is compact Hausdorff and \( S^n \) is Hausdorff the induced map is a homeomorphism.

1.7.6 Suppose that \( G \) is a topological group and \( H \) is a discrete subgroup. We will first show that \( \pi : G \to G/H \) is a covering space, where \( \pi(g) = gH \). Define a map \( r(g_1, g_2) = g_1^{-1}g_2 \). Because \( G \) is a topological group this is a continuous map. Choose a neighborhood \( U \) of the identity \( e \) in \( G \) so that \( U \cap H = \{e\} \). Since \( r \) is continuous \( r^{-1}(U) \) is open and since \( (e,e) \in r^{-1}(U) \) we can find open neighborhoods \( V_1 \) and \( V_2 \) of \( e \) in \( G \) so that \( V_1 \times V_2 \subseteq r^{-1}(U) \). Let \( V = V_1 \cap V_2 \). Then \( V \) is a neighborhood of \( e \) and if \( g_1 \) and \( g_2 \) are both in \( V \) then \( g_1^{-1}g_2 \in U \). I claim that \( gVH \) is an evenly covered neighborhood of the point \( gH \in G/H \). Clearly,

\[
\pi^{-1}gVH = \bigcup_{h \in H} gVh, \tag{1}
\]

Suppose that,

\[
gVh_1 \cap gVh_2 \neq \emptyset.
\]

Then for some \( g_j \in V \) we have,

\[
gg_1h_1 = gg_2h_2,
\]

which implies,

\[
U \ni g_1^{-1}g_2 = h_2h_1^{-1} \in H.
\]

Thus \( h_1 = h_2 \) and we see that (1) is a disjoint union. It is clear that \( \pi \) maps the open neighborhood \( gVh \) of \( gh \) in \( G \) bijectively on \( gVH \). To see that \( \pi \) is a homeomorphism it suffices to check that \( \pi \) is an open map. Suppose that \( W \) is open in \( G \) then

\[
\pi^{-1}(\pi W) = \pi^{-1}(WH) = \bigcup_{h \in H} Wh.
\]

Since \( G \) is a topological group \( Wh \) is open, hence the union in the preceeding equation is also open and so by definition \( \pi W \) is open. This shows that \( \pi \) is a covering space map.

Next we want to see that the deck transformations are all given by right translation by an element in \( H \). Define,

\[
\varphi_h(g) = gh^{-1}.
\]

It is simple to check that \( \varphi_h \) is a deck transformation and that \( \varphi_h(\varphi_{h'}(g)) = gh'^{-1}h^{-1} = g(hh')^{-1} = \varphi_{hh'}(g) \).

Now suppose that \( \varphi \) is a deck transformation. Then,

\[
\varphi(g)H = gH.
\]

This implies that \( g^{-1}\varphi(g) \) takes values in the subgroup \( H \). Since \( g \to g^{-1}\varphi(g) \) is continous, \( G \) is connected, and \( H \) has the discrete topology this implies that this map is constant. Thus there exists an \( h \) in \( H \) so that,

\[
\varphi(g) = gh,
\]

so \( \varphi = \varphi_{ha^{-1}} \).
1.7.12 If the group of covering transformations, \( \Gamma \), acts transitively on one fiber of a covering space \( p : Y \to X \) with a path connected base then \( \Gamma \) acts transitively in all fibers.

Proof: Suppose that \( \Gamma \) acts transitively in \( p^{-1}(x_0) \) for some \( x_0 \in X \). Let \( x_1 \in X \) denote a point in \( X \) and connect \( x_1 \) to \( x_0 \) by a path \( \sigma \) in \( X \). Suppose that \( u_1 \) and \( v_1 \) are in the fiber \( p^{-1}(x_1) \). Lift \( \sigma \) to a path \( \sigma_u \) in \( Y \) starting at \( u_1 \). Lift \( \sigma \) to a path \( \sigma_v \) starting at \( v_1 \). Let \( v_0 \in p^{-1}(x_0) \) denote the endpoint of \( \sigma_u \). Let \( u_0 \in p^{-1}(x_0) \) denote the endpoint of \( \sigma_v \). By hypothesis there exists a deck transformation \( \varphi \) so that \( \varphi(u_0) = v_0 \). Since a lift of \( \sigma \) is uniquely determined by either its initial or final endpoint it is clear that \( \varphi(\sigma_u) = \sigma_v \). But this implies that for the initial points \( \varphi(u_1) = v_1 \). Since \( u_1 \) and \( v_1 \) are arbitrary points in \( p^{-1}(x_1) \) it follows that \( \Gamma \) is transitive in \( p^{-1}(x_1) \).