Vector Bundles

A rank $k$ real vector bundle is a triple $(E, \pi, B)$ consisting of two Hausdorff spaces $E$ and $B$ and a continuous surjection,

$$\pi : E \rightarrow B,$$

which satisfies two additional conditions,

(a) For each $p \in B$ the inverse image $\pi^{-1}(p)$ is a $k$ dimensional real vector space.

(b) $B$ is covered by open sets $U_\alpha$ so that $\pi^{-1}(U_\alpha)$ is homeomorphic with $U_\alpha \times \mathbb{R}^k$ in a special way. There exists a homeomorphism $\varphi_\alpha$,

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k,$$

so that for $\pi(v) = p \in U_\alpha$,

$$\varphi_\alpha(v) = (p, \varphi^p_\alpha(v)),$$

where $\varphi^p_\alpha : \pi^{-1}(p) \rightarrow \mathbb{R}^k$ is a linear isomorphism.

The maps $\varphi_\alpha$ are called local trivializations of the vector bundle $E$. For any space $B$ we can define $E = B \times \mathbb{R}^k$ with,

$$\pi(b, v) = b.$$

This makes $B \times \mathbb{R}^k$ into a vector bundle over $B$ in a natural way and this bundle is called a trivial bundle. By definition vector bundles are always locally trivial, but they may not “look like” $B \times \mathbb{R}^k$ globally (we will spell this out in more detail later on).

It is useful to look at what happens when you change trivializations in a vector bundle. Suppose that $v \in E$ with $\pi(v) = p \in U_\alpha \cap U_\beta$. Then the map which takes $(p, \varphi^p_\alpha(v))$ to $(p, \varphi^p_\beta(v))$ is,

$$(p, u) \rightarrow (p, \varphi^p_\beta(\varphi^p_\alpha^{-1}(u))),$$

where $\varphi^p_\beta = (\varphi^p_\alpha)^{-1}$ and,

$$U_\alpha \cap U_\beta \ni p \rightarrow \varphi^p_\beta \in \text{GL}(n, \mathbb{R}).$$

The maps $p \rightarrow \varphi^p_\beta$ are called the transition maps for the bundle $E$. Transition maps which satisfy appropriate compatibility conditions can be used to define vector bundles.

The first example of a vector bundle for us is the tangent bundle of a manifold $M$, written $TM$. Putting topology aside for the moment, the total space $TM$ is the disjoint union of all the tangent spaces to $M$. That is,

$$TM = \bigcup_{p \in M} T_p M.$$

Although it is somewhat redundant it will be convenient to label the points in $TM$ as pairs $(p, v)$ with $v \in T_p M$ (the pair notation tells you $v \in T_p M$ without having to specify a base point for $v$ separately). The projection $\pi$ from $TM$ to $M$ is then,

$$\pi(p, v) = p.$$

If $x^\alpha$ is a coordinate on $U_\alpha \subset M$ then we can define a local trivialization for $\pi^{-1}(U_\alpha)$ by,

$$\pi^{-1}(U_\alpha) \ni (p, v) \rightarrow \varphi_\alpha(p, v) := (p, dx^\alpha_{p}(v)) \in U_\alpha \times \mathbb{R}^n.$$

Of course $dx^\alpha_p : T_p M \rightarrow \mathbb{R}^n$ is a linear map, so apart from the topology we have the appropriate sort of trivializations. Note that if $v \in T_p M$ and,

$$v = \sum_{j=1}^{n} v_j \left( \frac{\partial}{\partial x_j^p} \right)_{p},$$

then,

$$dx^\alpha_p(v) = (v_1, v_2, \ldots, v_n).$$
Thus $dx^\mu_p(v)$ just gives the coordinates of $v$ relative to the basis $\left\{ \left( \frac{\partial}{\partial x^\mu} \right)_p \right\}$ for $T_pM$.

Intuitively, it seems as if one could use the maps $\varphi_\alpha$ to pull back the product topology on $U_\alpha \times \mathbb{R}^n$ and get a topology on $TM$ (topologies are determined by the open neighborhoods of points). We will now examine a little more carefully how to do this.

Suppose that $X$ is a set which is the union of subsets $X_\alpha$. Suppose that for each $\alpha$ there are bijections $\varphi_\alpha : X_\alpha \to Y_\alpha$, where $Y_\alpha$ is a topological space. We can identify a topology $\mathcal{T}(X_\alpha)$ by saying that $\mathcal{O}$ is in $\mathcal{T}(X_\alpha)$ if and only if $\varphi_\alpha(\mathcal{O})$ is open in $Y_\alpha$. Suppose also that $X_\alpha \cap X_\beta$ is in $\mathcal{T}(X_\alpha)$ for all $\alpha$ and $\beta$ and that $\varphi_\alpha \varphi_\beta^{-1} : \varphi_\beta(X_\alpha \cap X_\beta) \to \varphi_\alpha(X_\alpha \cap X_\beta)$ is a homeomorphism for each $\alpha$ and $\beta$ with $X_\alpha \cap X_\beta$ non-empty. Define a topology on $X$ by saying that $\mathcal{O} \in \mathcal{T}(X)$ if and only if $\mathcal{O} \cap X_\alpha \in \mathcal{T}(X_\alpha)$ for all $\alpha$ (you might want to check that this defines a topology on $X$). The important property of this topology that will keep us from getting into trouble is that if $\mathcal{O}$ is open in $\mathcal{T}(X_\alpha)$ then $\mathcal{O}$ is open in $\mathcal{T}(X)$. This says that the topology induced in $\mathcal{T}(X_\alpha)$ by $\mathcal{T}(X)$ is just $\mathcal{T}(X_\alpha)$. In order to see this suppose that $\mathcal{O} \in \mathcal{T}(X_\alpha)$. Then,

$$\varphi_\beta(\mathcal{O} \cap X_\beta) = \varphi_\beta \varphi_\alpha^{-1}(\varphi_\alpha(\mathcal{O} \cap X_\alpha \cap X_\beta)).$$

But

$$\varphi_\alpha(\mathcal{O} \cap X_\alpha \cap X_\beta) = \varphi_\alpha(\mathcal{O}) \cap \varphi_\alpha(X_\alpha \cap X_\beta).$$

This last intersection of open sets in $Y_\alpha$ is open and since $\varphi_\beta \varphi_\alpha^{-1}$ is a homeomorphism we see that $\varphi_\beta(\mathcal{O} \cap X_\beta)$ is open as well.

Let's check to see how this construction applies to give a topology for $TM$. The open subsets $U_\alpha$ of $B$ have topologies that arise from the topology of $B$. We give $Y_\alpha = U_\alpha \times \mathbb{R}^n$ the product topology. It is easy to see that $\varphi_\alpha(p,v) = (p,dx^\alpha_p(v))$ is a bijection of $X_\alpha = \pi^{-1}(U_\alpha)$ with $Y_\alpha$. It is also easy to check that,

$$\varphi_\alpha(X_\alpha \cap X_\beta) = U_\alpha \cap U_\beta \times \mathbb{R}^n,$$

which is open in $Y_\alpha$ since $U_\alpha \cap U_\beta$ is open in $U_\alpha$. Finally the change of trivializations (we write $x_\alpha = x^\alpha$ to avoid a collision of superscripts),

$$\varphi_\alpha \varphi_\beta^{-1}(p,v) = (p,d(x_\alpha x_\beta^{-1})_{x_\beta(p)}(v)), \quad \text{for } v \in \mathbb{R}^n,$$

is continuous since $x_\alpha x_\beta^{-1}$ is $C^1$ with a $C^1$ inverse. With this topology all the conditions for $TM$ to be a vector bundle are verified except perhaps the condition that $TM$ has a Hausdorff topology. To see that the topology is Hausdorff suppose that $(p,v)$ and $(p',v')$ are in $TM$. If $p$ and $p'$ are distinct, then because the base $M$ is Hausdorff, there exist disjoint open neighborhoods $U_\alpha$ and $U_\beta$ so that $p \in U_\alpha$ and $p' \in U_\beta$. Then $\pi^{-1}(U_\alpha)$ and $\pi^{-1}(U_\beta)$ are disjoint neighborhoods of $(p,v)$ and $(p',v')$. Now suppose that we want to separate points $(p,v)$ and $(p,v')$ with $v \neq v'$. If $p \in U_\alpha$ the domain of a chart $x^\alpha$ we can find open sets $V$ and $V'$ in $\mathbb{R}^n$ so that $dx^\alpha_p(V) \subset V$ and $dx^\alpha_{p'}(V') \subset V'$. Then $\varphi_\alpha^{-1}(U_\alpha \times V)$ and $\varphi_\alpha^{-1}(U_\alpha \times V')$ are open sets in $TM$ that separate $(p,v)$ and $(p,v')$. This finishes the proof that the topology of $TM$ is Hausdorff.

I've made a bit of a production out of this since the same construction works to produce a topology on a space which is locally Euclidean but in general the topology is not Hausdorff. The union of the open interval $(-1,1)$ with the half open interval $[2,3)$ can be given a topology in which the (small) open neighborhoods of all points are the usual ones except at the point $2$ which is declared to have open neighborhoods of the form $(-\epsilon, 0] \cup [2,2+\epsilon)$. Each point has a neighborhood which is homeomorphic to an open interval but the points 0 and 2 cannot be separated by disjoint open sets. For many local constructions the non Hausdorff character is not important but when we later want to construct metrics and use partitions of unity it will be necessary to restrict our attention to Hausdorff spaces (with some further restrictions as well).

If $(E, \pi, B)$ is a vector bundle and $U$ is a subset of $B$ then a section of $E$ over $U$ is a continuous map, $\sigma : U \to E$, such that $\pi(\sigma(p)) = p$. That is $\sigma(p) \in \pi^{-1}(p)$. Sections of the tangent bundle of a manifold are called vector fields.
The tangent bundle $TM$ is a manifold in its own right. I will leave it to you to check that if $(x^\alpha, U_\alpha)$ is a chart for $B$ then $(p, v) \rightarrow (x^\alpha(p), dx^\alpha_j(v))$ is a chart on $\pi^{-1}(U_\alpha)$ for $TM$. If $U$ is an open subset of $M$ we will say that a section, $\sigma$, of $TM$ over $U$ is $C^r$ provided the map,

$$\sigma : U \rightarrow TM,$$

is $C^r$ as a map of manifolds. So we can talk about $C^r$ vector fields. You should check that $\sigma$ is a $C^r$ vector field provided that in each chart $x^\alpha$ the coordinate functions $p \rightarrow \sigma_j(p)$ in the representation,

$$\sigma_p = \sum_{j=1}^{n} \sigma_j(p) \left( \frac{\partial}{\partial x^\alpha_j} \right)_p,$$

are $C^r$ functions on $U_\alpha$.

Let $T^*_p M$ denote the dual of the tangent space to $M$ at $p$ (the space of linear functionals on $T_p M$). The dual spaces $T^*_p M$ “glue together” into a vector bundle $T^* M$, called the cotangent bundle of $M$, which is given as the disjoint union,

$$T^* M = \sqcup_{p \in M} T^*_p M.$$

Trivializations are obtained in the following manner. Suppose $(x^\alpha, U_\alpha)$ is a chart for $M$ and suppose $p \in U_\alpha$ and $v \in T^*_p M$. Define,

$$\pi^{-1}(U_\alpha) \ni (p, v) \rightarrow \varphi_\alpha(p, v) = \left( p, v \left( \frac{\partial}{\partial x^1} \right)_p, \ldots, v \left( \frac{\partial}{\partial x^n} \right)_p \right).$$

Note that if,

$$v = \sum_{j=1}^{n} v_j dx^\alpha_j(p),$$

then,

$$\varphi_\alpha(p, v) = (p, v_1, v_2, \ldots, v_n).$$

This is just a reflection of the fact that $\left\{ \left( \frac{\partial}{\partial x^1} \right)_p, \ldots, \left( \frac{\partial}{\partial x^n} \right)_p \right\}$ are dual basis (each is the coordinate function for the other). You should check that the change of trivialization is given by $(x_\alpha = x^\alpha$ again)

$$\varphi_{\alpha \beta}(p, v) = (p, d(x_\alpha x^{-1}_\beta)_{x_\alpha}(p), v),$$

where $-T$ refers to the “inverse transpose”. Sections of $T^* M$ are called one forms and as in the case of $TM$ there is a natural manifold structure for $T^* M$ which allows one to talk about $C^r$ one forms. As above you should confirm that $\sigma$ is a $C^r$ one form provided that in each chart $x^\alpha$ the coordinate functions $p \rightarrow \sigma_j(p)$ in the representation,

$$\sigma_p = \sum_{j=1}^{n} \sigma_j(p) dx^\alpha_j,$$

are $C^r$ functions on $U_\alpha$.

Observe that if $f : M \rightarrow \mathbb{R}$ is a smooth function then $df_p \in T^*_p M$ and it is easy to see that $df$ is actually a smooth section of $T^* M$.

If $M$ and $N$ are smooth manifolds and $f : M \rightarrow N$ is a smooth map then we define $f_* : TM \rightarrow TN$ as follows,

$$f_*(p, v) = (f(p), Df(p)(v)).$$

There is no corresponding map induced by $f$ on $T^* M$ and $T^* N$ but if $\sigma$ is a smooth section of $T^* N$ then it is easy to see that,

$$(f^* \sigma)_p := \sigma_{f(p)} \circ df_p,$$

is a section of $T^* M$. The section $f^* \sigma$ is called the pull back of the section $\sigma$. 3