Solutions to Homework 1, Math 534a

1. Let \( X \) and \( Y \) denote the \( x \) and \( y \) axes in \( \mathbb{R}^2 \). If \( X \cup Y \) is locally Euclidean then for a sufficiently small ball, \( B \), about \( (0,0) \) there exists a homeomorphism,

\[ \varphi : X \cup Y \cap B \rightarrow U, \]

where \( U \) is connected open subset of \( \mathbb{R}^n \). The complement of \( (0,0) \) in \( X \cup Y \cap B \) has four connected components and the complement of \( \varphi(0,0) \) in \( U \) has \( n \geq 2 \) and \( 2 \) components for \( n = 1 \). The homeomorphic image of a space with four connected components must have \( 4 \) connected components. Thus no such homeomorphism can exist.

2. Write \( (x, y) \in \mathbb{R}^{n+1} \) with \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R} \). The line through \( e_{n+1} = (0, 1) \) and \( p = (x, y) \in S^n \) has the parametric representation,

\[ (1 - t)e_{n+1} + t(x, y). \]

The \( n + 1 \) coordinate on this line vanishes for \( t = \frac{1}{1 - y} \) so the image of stereographic projection from the north pole is,

\[ \varphi(p) = \frac{x}{1 - y} \in \mathbb{R}^n. \]

The line through \( e_{n+1} \) and \( (x, 0) \) is,

\[ (1 - s)e_{n+1} + s(x, 0). \]

This is on the unit sphere if,

\[ (1 - s)^2 + s^2|x|^2 = 1, \text{ or } s = \frac{2}{1 + |x|^2}. \]

(the other root \( s = 0 \) just gives the north pole). Thus,

\[ \varphi^{-1}(x, 0) = \left( \frac{2x}{1 + |x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1} \right). \]

It is easy to see that both \( \varphi : S^{n+1} \setminus \{e_{n+1}\} \rightarrow \mathbb{R}^n \) and \( \varphi^{-1} \) are continuous, so \( \varphi \) is a homeomorphism. Stereographic projection from the south pole \( -e_{n+1} \) is,

\[ \psi(p) = \frac{x}{1 + y}. \]

Composing this with \( \varphi^{-1} \) we find,

\[ \psi\varphi^{-1}(x) = \frac{2x}{1 + |x|^2}, \quad \frac{1 + |x|^2}{2|x|^2} = \frac{x}{|x|^2}. \]

3. Let \( A \) denote the antipodal map, \( Ax = -x \), acting on \( S^n \). Then \( A \) is continuous and since it is its own inverse it is a homeomorphism of \( S^n \). The topology of \( P^n \) is defined so that the map \( \pi : S^n \rightarrow P^n \) is continuous. It is also an open map. To see this, suppose that \( U \) is open in \( S^n \). Then \( \pi(U) \) is open in \( P^n \) if and only if,

\[ \pi^{-1}(\pi(U)) = U \cup AU, \]

is open in \( S^n \). But \( AU \) is open since \( A \) is a homeomorphism. Thus \( \pi \) is an open map. Let \( H_k \) denote the closed hyperplane \( x_k = 0 \) in \( \mathbb{R}^{n+1} \). Then \( S^n \setminus H_k = V_k^+ \cup V_k^- \) where,

\[ V_k^\pm = \{ x \in S^n : \pm x_k > 0 \}. \]

The map \( \pi : V_k^+ \rightarrow \pi V_k^+ \) is a continuous bijective open map and hence is a homeomorphism. The open sets \( \pi V_k^\pm \) for \( k = 1, \ldots, n+1 \), cover \( P^n \) since any point not in the union would have to have vanishing coordinates.
for all \( k \). To find a homeomorphism of \( \pi V_k^+ \) with an open set in \( \mathbb{R}^n \) it suffices to find a homeomorphism of \( V_k^+ \) with and open subset of \( \mathbb{R}^n \). Such a map is given by,

\[
V_k^+ \ni x \mapsto (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1})
\]

with inverse,

\[
D^n \ni x \mapsto (x_1, \ldots, x_{k-1}, \sqrt{1 - |x|^2}, x_k, \ldots, x_n),
\]

where the square root is the positive choice. Thus \( P^n \) is locally Euclidean. Since every open set in \( P^n \) is \( \pi(U) \) for an open set \( U \) in \( S^n \) and \( S^n \) has a countable base it follows that \( P^n \) also has a countable base (just the image under \( \pi \) of the base in \( S^n \)). To see that \( P^n \) is Hausdorff suppose that \( p = \{ x, -x \} \) and \( q = \{ y, -y \} \) are two distinct points in \( P^n \). Let \( \epsilon > 0 \) be chosen less than,

\[
\frac{1}{2} \min\{|x - y|, |x + y|\},
\]

and define, \( B(x, \epsilon) = \{ u \in S^n : |u - x| < \epsilon \} \). Then the open sets \( \pi B(x, \epsilon) \) and \( \pi B(y, \epsilon) \) don’t intersect and so \( P^n \) is Hausdorff.

4. The map \( F(t) = \exp(2\pi it) \) from \([0,1]\) to \( S^1 \) is clearly continuous, and it induces a bijective map from \([0,1]/\sim\) to \( S^1 \) which we denote by \( f \). Evidently \( f \pi = F \). Thus \( f \) is continuous since \( f^{-1}(U) \) is open iff \( \pi^{-1}(f^{-1}(U)) = F^{-1}(U) \) is open and \( F \) is continuous. Since \([0,1]/\sim\) is the continuous image of a compact space it is compact. It is easy to see how to separate points in \([0,1]/\sim\) so this space is Hausdorff. Since \( f \) is continuous and bijective, \([0,1]/\sim\) is compact and Hausdorff, and \( S^1 \) is Hausdorff it follows that \( f \) is a homeomorphism.