Comments on Homework 4 Math 534

2.3.19 Suppose $f$ is a function on $S^n$ with a smooth extension $\tilde{f}$ to a neighborhood of $S^n$. Show by example that $df$ may depend on the extension but $df|_{T_p S^n}$ does not depend on the extension. $\tilde{f}_1(x) = |x|^2$ and $\tilde{f}_2(x) = 2 - |x|^2$ both extend the constant function $f(x) = 1$ on $S^n$ but obviously have different derivatives on $\mathbb{R}^{n+1}$. Now suppose that $\tilde{f}$ is a smooth extension of a function $f$ defined on $S^n$. Then the restriction of $df$ to $T_p S^n$ is determined by what $\tilde{f}$ does to curves which live on $S^n$. But the action of $\tilde{f}$ on a curve which lives on $S^n$ is the same as the action of $f$ and hence is independent of the extension.

2.4.13 Use 2.4.11 to show that the unit sphere $S^n$ is an imbedded submanifold of $\mathbb{R}^{n+1}$.

Let $f(x) = x_1^2 + \cdots + x_{n+1}^2$. To show that $S^n$ is a an imbedded submanifold we must show that each point $p \in S^n$ has a neighborhood in the *relative topology* which is homeomorphic to an open subset of $\mathbb{R}^n$. Suppose that $p \in S^n$. For some integer $k$ there must be a coordinate $p_k$ which is not 0. Thus $\partial f/\partial x_k |_p = 2p_k \neq 0$. Therefore 2.4.11 implies that there is an open ball $B$ in $\mathbb{R}^{n+1}$ centered at $p$ and a differentiable map $g$ defined on an open set $U$ in $\mathbb{R}^n$ so that the map,

$$U \ni x \rightarrow G(x) := (x_1, \ldots, x_k, g(x), x_{k+1}, \ldots, x_n) \in S^n \cap B,$$

is bijective. To see that this map is a homeomorphism we need only check that the map which takes $G(x)$ to $x$ is continuous. This is obvious, since this map is just the restriction to $S^n$ of projection on all but the $k^{th}$ coordinate in $\mathbb{R}^{n+1}$.

2.5.9 Define $\Phi(Y) = Y^T Y$ and consider $\Phi$ as a map from $GL(n)$ to $GL(n)$. Suppose that $g \in GL(n)$. We want to calculate the rank of $d\Phi_g : T_g GL(n) \rightarrow T_{\Phi(g)} GL(n)$. Every tangent vector in $T_g GL(n)$ can be realized as an infinitesimal curve,

$$(I + tA)g,$$

for some $A \in \mathcal{M}(n)$, the $n \times n$ matrices. This infinitesimal curve is carried by $\Phi$ to the infinitesimal curve,

$$(g^T (I + tA^T)(I + tA)g).$$

Identifying these infinitesimal curves with their tangents in Euclidean space we see that,

$$d\Phi_g(A)g = g^T (A^T + A)g.$$

The dimension of the null space of this map is clearly the dimension of the skew symmetric matrices (the right hand side is 0 if and only if $A^T + A = 0$) or $\frac{n(n-1)}{2}$. Thus the rank (the dimension of the range) is $n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

The constant rank theorem applies to show that $\Phi^{-1}(I) = O(n)$ is submanifold of $GL(n)$. To see that $O(n)$ is compact we need only see that it is a closed and bounded subset of $\mathcal{M}(n)$. It is closed since it is a level set for the continuous function $\Phi$ and it is bounded since $\text{Tr}(X^TY)$ is an inner product on $\mathcal{M}(n)$ and $\text{Tr}(XX^T) = n$ for $X \in O(n)$.

2.5.14 Show that an $r$ dimensional submanifold $N$ is parallelizable iff there exist $r$ smooth vector fields $v_1, \ldots, v_r$ on $N$ which are linearly independent at each point $p \in N$. Show $\text{SL}(n)$ and $O(n)$ are parallelizable.

Suppose that $N$ has $r$ smooth vector fields $v_j(p)$ $j = 1, \ldots, r$ which are linearly independent at each point $p \in N$. This means that there exists an open neighborhood $U$ of $N$ in $\mathbb{R}^n$ and smooth maps,

$$\tilde{v}_j : U \rightarrow \mathbb{R}^n$$

so that $\tilde{v}_j$ restricted to $N$ is $v_j$. Suppose $p \in N$. Then since $\{v_1(p), \ldots, v_k(p)\}$ are linearly independent there exist vectors $v_{k+1}(p), \ldots, v_r(p)$ so that $\{v_1(p), \ldots, v_n(p)\}$ is a basis for $\mathbb{R}^n$. Choose an open neighborhood $U_p$ of $p$ in $\mathbb{R}^n$ so that for all $q \in U_p$,

$$\det[\tilde{v}_1(q), \ldots, \tilde{v}_k(q), v_{k+1}(p), \ldots, v_n(p)] \neq 0.$$
Evidently,

\[ \{ \tilde{v}_1(q), \ldots, \tilde{v}_k(q), v_{k+1}(p), \ldots, v_n(p) \}, \]

is a basis for \( \mathbb{R}^n \) for all \( q \in U_p \). Now suppose that \( (p, X) \in TN \). We define a map from \( TN \) to \( N \times \mathbb{R}^k \) by,

\[ \varphi(p, X) = (p, x_1(p), \ldots, x_k(p)), \]

(1)

where,

\[ X = \sum_{j=1}^{k} x_j(p) v_j(p). \]

That is, \( x_j(p) \) is just the \( j^{th} \) coordinate of \( X \) with respect to the basis \( \{ v_1(p), \ldots, v_k(p) \} \). The map (1) is clearly well defined. It remains to show that \( \varphi \) is smooth with a smooth inverse. This is a local property and so we show that for any \( p \in N \) there is a smooth extension of \( \varphi \) to \( U_p \times \mathbb{R}^n \). For \( q \in U_p \) and \( X \in \mathbb{R}^n \) define,

\[ \bar{\varphi}(q, X) = (q, x_1(q), \ldots, x_n(q)), \]

where,

\[ X = \sum_{j=1}^{k} x_j(q) \bar{v}_j(q) + \sum_{j=k+1}^{n} x_j(q) v_j(p), \]

so that \( x_j(q) \) is just the \( j^{th} \) coordinate for \( X \) with respect to the basis \( \{ \tilde{v}_1(q), \ldots, \tilde{v}_k(q), v_{k+1}(p), \ldots, v_n(p) \} \) for \( \mathbb{R}^n \). It is obvious that \( \bar{\varphi} \) is an extension of \( \varphi \).

Define the \( n \times n \) matrix \( V(q) \) by,

\[ V(q) = [\tilde{v}_1(q), \ldots, \tilde{v}_k(q), v_{k+1}(p), \ldots, v_n(p)]. \]

Then the coordinates \( x_j(q) \) of the column vector \( x(q) \) are defined by the matrix relation,

\[ V(q)x(q) = X. \]

Thus

\[ x(q) = V(q)^{-1}X. \]

Cramer’s rule shows that \( V(q)^{-1} \) is smooth in \( q \) if \( V(q) \) is smooth. The inverse map \( \bar{\varphi}^{-1} \) is,

\[ \bar{\varphi}^{-1}(q, x) = (q, \sum_{j=1}^{k} x_j \tilde{v}_j(q) + \sum_{j=k+1}^{n} x_j v_j(p)). \]

This is smooth since \( \tilde{v}_j(q) \) are assumed to be smooth and this finishes the proof.

We will show that \( O(n) \) is parallelizable, the proof for \( SL(n) \) is similar. For each \( X \in \mathcal{M}(n) \) define a vector field \( v_X \) on \( GL(n) \) by,

\[ v_X(g) = Xg, \quad \text{for } g \in GL(n), \]

where \( Xg \) is the tangent to the infinitesimal curve \( \langle (I + tX)g \rangle \). For fixed \( X \) this is clearly a smooth function of \( g \) (which is just contained in an open subset of \( \mathbb{R}^{n^2} \)). Let \( X_1, \ldots, X_k \) be a basis for the skew symmetric matrices \( (k = \frac{n(n-1)}{2}) \) and define,

\[ v_j(g) = V_{X_j}(g) \quad \text{for } g \in O(n). \]

It is easy to check that \( v_j \) is smooth since it is the restriction of \( V_{X_j} \) from \( GL(n) \) to \( O(n) \) and it is easy to see that the sections \( v_j \) are linearly independent at each point since any linear relation among the \( X_jg \) would lead to the same linear relation among the \( X_j \) (since \( g \) is invertible).
2.5.15 Suppose that $N$ is a smooth $r$ dimensional submanifold of $\mathbb{R}^n$ and $p \in N$. We want to show that there is a neighborhood $U$ of $p \in N$ so that $TU \subset T N$ is trivial. We know that $p$ has a neighborhood $V$ in $\mathbb{R}^n$ with a diffeomorphism $\varphi : V \to B$ where $B$ is an open ball in $\mathbb{R}^n$ with $\varphi(V \cap N) = B \cap \mathbb{R}^k$. The map,

$$(q,v) \to (q, d\varphi_q(v)), \text{ for } q \in V \cap N, v \in T_q V \cap N,$$

is easily seen to be smooth (it is the restriction of the same map defined on $T(V) \simeq V \times \mathbb{R}^n$) and is a trivialization of $TU$ for $U = V \cap N$. The modified map,

$$\varphi_*(q,v) = (\varphi(q), d\varphi_q(v)),$$

is a diffeomorphism that identifies $TU$ with $B \cap \mathbb{R}^k \times \mathbb{R}^k \subset \mathbb{R}^n \times \mathbb{R}^n$ and so serves as a chart for points in $TU$. 