UNVR 397A – Math Sense
A Brief Introduction to Logic

Many of the problems that students encounter in mathematics stem from misconceptions about logic. This worksheet is designed to briefly introduce you to some principles of logic that frequently confuse students, allow you to work through some exercises regarding logic, and most importantly, to give you a chance to work on strategies to help students who are struggling with the basic principles of logic.

Part 1. If-Then Statements

One of the most common statements that we encounter in mathematics (and in everyday life) is the if-then statement. For example, consider the statement below:

Statement \( P \): “If it is raining, then there are clouds in the sky.”

Clearly, this is a true statement. One way to represent it graphically is to use a Venn Diagram. If we let \( R \) stand for the statement “it is raining” and \( C \) stand for the statement “there are clouds in the sky,” we can draw a circle for each statement. Think of the circle \( R \) as containing all days on which it is raining and the circle \( C \) as containing all days on which it is cloudy, as shown below:

\[
\text{\includegraphics{venn_diagram.png}}
\]

Note that the \( R \) circle has been drawn inside of the \( C \) circle to communicate the meaning of our original if-then statement: that any rainy day must be included among the cloudy days. Think of the Venn diagram above as a dartboard; each dart that you throw illustrates one possible outcome, as shown below.

**Figure 1.** Illustration of the implication statement, “If it is raining, then there are clouds in the sky.”

A cloudy day on which it rains.  A cloudy day on which it does not rain.  A day on which it is not cloudy and does not rain.

Thinking in terms of darts, the implication illustrated in the above figure is equivalent to saying, “If a dart hits inside of \( R \), then it hits inside of \( C \).”

One of the nice features of a Venn Diagram is that it allows us to take an if-then statement and analyze related if-then statements. For example, consider the statements below.

Original Statement \( P \): If it is raining, then there are clouds in the sky.

Statement \( Q \): If there are clouds in the sky, then it is raining.

Statement \( R \): If there are not clouds in the sky, then it is not raining.

Are either of the two new statements \( Q \) or \( R \) equivalent to \( P \)? Let us analyze this question in terms of our Venn Diagrams from Figure 1. In terms of darts, statement \( Q \) says: “If a dart hits inside circle \( C \), then it hits inside \( R \).” This is clearly not the case, so we conclude that \( Q \) and \( P \) are not equivalent statements. On the other hand, statement \( R \) says: “If a dart hits outside of \( C \), then it hits outside of \( R \).” We can see that this statement is consistent with Figure 1 and therefore conclude that \( P \) and \( R \) are equivalent statements.

In the language of logic, \( Q \) is called the converse of \( P \), and \( R \) is called the contrapositive of \( P \). We have therefore illustrated that an if-then statement is logically equivalent to its contrapositive but not logically equivalent to its converse.
Part 2. Exercises

1. Draw two Venn diagrams, one illustrating the statement $P_1$: “If we have a Math Sense meeting, then it is Thursday,” and one illustrating the statement $P_2$: “If it is Thursday, then we have a Math Sense meeting.” Which of these two propositions says the same thing as “If we don’t have a Math Sense meeting, then it is not Thursday?”

2. Let $P$ be the statement shown below.

   $P$: “If a jelly bean is buttered popcorn flavored, then Jerry won’t eat it.”

   Which of the following statements say the same thing as $P$?

   (a) Jerry won’t eat a jelly bean whenever it is buttered popcorn flavored.
   (b) Jerry will eat a jelly bean if it is not buttered popcorn flavored.
   (c) Jerry will eat a jelly bean only if it is not buttered popcorn flavored.
   (d) Jerry will eat a jelly bean whenever it is not buttered popcorn flavored.
   (e) If a jelly bean is not buttered popcorn flavored, then Jerry will eat it.
   (f) In order for Jerry to eat a jelly bean, it is necessary that it not be buttered popcorn flavored.

3. Consider the following (true) theorem that is often taught to students in a college algebra class:

   **Theorem.** If $p(x)$ is a polynomial of odd degree, then the equation $p(x) = 0$ has a real root.

   In each of the following situations below, a student applies the theorem shown above. Your job is to decide whether or not the student has correctly used the theorem. If the argument is incorrect, find the error, illustrate to the student that the answer doesn’t make sense, and then help clear up the logical misconception that produced the error.

   (a) Student Observation: ‘‘The function $f(x) = x^2 + 1$ never crosses the $x$-axis and therefore has no roots. Aha! Now I see what this theorem is saying. My polynomial has even degree, so it and all other polynomials with even degree will not have real roots.’’
   (b) Student Observation: ‘‘Any polynomial whose graph never crosses the $x$ axis has to have even degree.’’
   (c) Student Concern: ‘‘I graphed the function $f(x) = x^3 - 2x^2 - x + 2$ on my calculator and noticed that it has three real roots, not just one. The theorem must be wrong!’’

4. One day, I walked into a restaurant and noticed the following sign displayed on their wall. (This is a true story, by the way.)

   ![NO SMOKING]

   I was amused by the use of the double negative, so I drew the sign on the board the next day for my students to see. In discussing the meaning of the sign, most of them agreed that the meaning of the sign as it was displayed was “Smoking is allowed.” Is this an accurate interpretation? Explain.

5. In each of the following situations, a student has made some sort of error in reasoning. Find the error, and think of a way to illustrate to help the student see why their statement is wrong.

   (a) Student Statement: ‘‘If a function is not even, it must be odd.’’
   (b) Student Statement: ‘‘I can see that the function $f(x) = (x - 1)^2$ is symmetric, so it must be even.
   (c) Student Statement: ‘‘The function $f(x) = x^3 + 1$ has only odd powers of $x$ in it, so it must be odd.’’